BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A Historical and Critical Review of Mathematical Science

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BULLETIN

OF THE

AMERICAN MATHEMATICAL SOCIETY

THE THIRTY-FIRST SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

A notable occasion in the annals of the Society was the holding of the thirty-first summer meeting and tenth colloquium of the Society at Cornell University, Ithaca, N.Y., from Tuesday to Saturday, September 8-12, 1925. For such a gathering the location was ideal; the local arrangements left nothing to be desired; there was a record attendance, and enthusiasm ran high.

The colloquium lectures, by Professors Eisenhart and Jackson, were delivered on Tuesday, Wednesday, and Friday afternoons, Thursday evening, and Saturday morning. On Thursday and Friday mornings the Society met in sections for the reading of papers, the sections of Geometry and of Point Sets and Foundations meeting on Thursday, and those of Analysis and of Algebra and Theory of Numbers on Friday. Sessions of the Mathematical Association of America were held on Tuesday and Wednesday mornings. The joint dinner of the Society and the Association, held at the Bank Restaurant in Ithaca on Wednesday evening, with Professor Slaught as toastmaster, was attended by one hundred eighty-five persons.

Many attending members recalled with pleasure the two earlier summer meetings of the Society held at Cornell, in 1901 and 1907. The Society held its Third Colloquium in connection with the 1901 meeting; on this occasion Professor

Oskar Bolza lectured on *The simplest type of problems in the calculus of variations*, and Professor E. W. Brown on *Modern methods of treating dynamical problems and in particular the problem of three bodies*. The 1925 colloquium should therefore be known as the Second Ithaca Colloquium. A special report of this colloquium by Professor Hildebrandt is given in the present issue of this BULLETIN on page 24.

The attending mathematicians and their friends were entertained at the Sage College dormitory of the University, and the meetings were held in the Baker Chemical Laboratory. On Tuesday evening President Farrand and the Mathematics Department welcomed the visitors at a reception in Sage College. Through the courtesy of the Automobile Club of Ithaca an excursion was made on Thursday afternoon to Enfield Glen, a very beautiful gorge near Ithaca. The Ithaca Country Club extended an invitation to visitors to make use of its facilities. Other enjoyable occasions were a luncheon at the home of Mrs. Owens for attending ladies, a reception at the Ithaca Country Club given by Mrs. Tanner, a visit to the Fuertes Observatory, and an organ recital by Professor Ballard of the department of Electrical Engineering. A hearty vote of thanks was passed to the various hosts and to the local members of the Committee on Arrangements, Professors Tanner, Gillespie, and Hurwitz.

One hundred twenty-two members of the Society heard the Colloquium lectures. Attendance at the sessions of the Society included the following one hundred fifty members:

C. R. Adams, R. B. Adams, Akers, W. E. Anderson, Archibald, Bacon, Bareis, E. R. Beckwith, Beisel, Bell, Bennett, Herman Betz, Birkhoff, Bliss, Boothroyd, H. S. Brown, Margaret Buchanan, W. G. Bullard, R. W. Burgess, Bussey, Callahan, B. H. Camp, A. D. Campbell, Carr, Carus, Carver, Coble, Teresa Cohen, J. T. Colpitts, Coolidge, L. P. Copeland, Court, Craig, Dale, Decker, Denton, Dowling, Dresden, Dunkel, Eiesland, Eisenhart, W. W. Elliott, G. C. Evans, G. W. Evans, Everett, Peter Field, W. B. Ford, Fort, Fry, Gaba, Gale, Garretson, D. C. Gillespie, Gilman, Glenn, J. W. Glover, M. C. Graustein, W. C. Graustein, Gravatt, L. M. Graves, E. R. Hedrick, Herr, Hildebrandt, Hille, Hollcroft, A. M. Howe, Hurwitz, Hutchinson, Ingraham, Dunham Jackson, B. W. Jones,

O. D. Kellogg, B. F. Kimball, Kuhn, Lambert, Lefschetz, Long, Lubben, J. V. McKelvey, M. M. McKelvey, MacColl, MacCreadie, MacDuffee, Maria, H. A. Merrill, Michal, E. A. Miller, G. A. Miller, Norman Miller, H. H. Mitchell, Molina, C. L. E. Moore, C. N. Moore, Morenus, Richard Morris, D. S. Morse, Marston Morse, F. H. Murray, Nassau, Olds, Olson, F. W. Owens, H. B. Owens, Packer, L. R. Perkins, Poritsky, Ranum, Rasor, R. G. D. Richardson, H. L. Rietz, Robinson, E. D. Roe, J. R. Roe, Rosenbaum, Schelkunoff, Seely, Sellew, Sharpe, Shewhart, Shohat, Slaught, Smail, C. E. Smith, H. F. Smith, Snedecor, Virgil Snyder, Stocker, Sullivan, Tamarkin, Tappan, J. H. Taylor, E. M. Thomas, J. M. Thomas, T. Y. Thomas, Trevor, B. M. Turner, Vandiver, Vivian, Waddell, G. W. Walker, Watkeys, H. E. Webb, Wedderburn, Weisner, Westfall, W. L. G. Williams, Williamson, E. W. Wilson, Yeaton.

The Assistant Secretary announced that the following persons and institutions had been elected to membership since the April meeting of the Council:

To sustaining membership:

International Life Insurance Company, St. Louis, Mo.;
Massachusetts Institute of Technology, Cambridge, Mass.;
Northwestern University, Evanston, Ill.;
Philadelphia Electric Company, Philadelphia, Pa.;
Western and Southern Life Insurance Company, Cincinnati, Ohio.

To ordinary membership:

Mr. Glenn Potter Aldrich, State University of Iowa;
Professor Mildred Allen, Mt. Holyoke College;
Mr. Ben Raymond Beisel, Cornell University;
Mr. Charles Franklin Bowles, South Dakota State School of Mines;
Dr. Gregory Breit, Department of Terrestial Magnetism, Carnegie Institution of Washington;
Mr. William Marshall Bullitt, Louisville, Ky.;
Mr. Robert Hamilton Coats, Dominion Bureau of Statistics, Ottawa, Canada;
Mr. Irving Smith Cranford, Columbia University;
Mr. George Adams Ellis, New York, N.Y.;
Mr. Hallett Barker Hammatt, Harvard University;
Mr. George Hartwell, Cheltenham Magnetic Observatory;
Dr. William Jackson Humphreys, United States Weather Bureau;
Mr. Wilmer Atkinson Jenkins, University of Michigan;
Professor Louis Vessot King, McGill University;
Mr. Alfred Korzybski, New York, N.Y.;
Mr. Silvio G. Lanzon, Toronto, Canada;
Mr. Lincoln LaPaz, Dartmouth College;
Mr. Morris Marden, Harvard University;
Mr. Tsao Matsumura, Miyazaki Normal School;

Vice-President Paul Vaughan Montgomery, Southland Life Insurance Company, Dallas, Tex.;
Mr. Lucius Terrell Moore, Johns Hopkins University;
Professor Harold R. Phalen, Armour Institute of Technology;
Mr. Boris Podolsky, Los Angeles, Calif.;
Mr. Elder Alexander Porter, Indianapolis Life Insurance Company;
Professor Adrien Pouliot, Laval University;
Mr. Henry William Raudenbush, Columbia University;
Mr. Pierre Gain Robinson, Iowa State College;
Professor Herbert E. Russell, University of Denver;
Miss Faith Saunders, Logan College;
Professor Albert Norman Shaw, McGill University;
Miss Mary Frazer Smith, Wellesley College;
Mr. Martin Joseph Spinks, Champion Bridge Company, Wilmington, Ohio;
Mr. Arthur Francis Chesterfield Stevenson, University of Toronto;
Professor Jacques Tamarkin, Dartmouth College;
Miss Frances Edget Thomas, Wells College;
Dr. Manuel Sandoval Vallarta, Massachusetts Institute of Technology;
Mr. Hubert Stanley Wall, University of Wisconsin;
Mr. Morgan Ward, California Institute of Technology;
Professor Walter F. Willcox, Cornell University;
Mr. Don M. Yost, California Institute of Technology;
Dr. Otto Julius Zobel, American Telephone and Telegraph Company, New York, N.Y.;

Nominees of the Aetna Life Insurance Company, Hartford, Conn.:
Paul Dorweiler, Morley I. Doxsie, Roland S. Haradon, Elton B. Hill, Walter S. Paine;

Nominees of the American Life Insurance Company, Detroit, Mich.:
William H. Brown, C. F. Cross, Walter H. Ekberg;

Nominees of the Detroit Life Insurance Company, Detroit, Mich.:
Henry R. Carstens, Clements H. Kettenhofen, Eva M. Roe, Margaret Toft, E. C. Wightman;

Nominees of Harvard University, Cambridge, Mass.:
Professors Harry E. Clifford, Hector J. Hughes, Theodore Lyman, Harlow Shapley, George F. Swain;

Nominee of the International Life Insurance Company, St. Louis, Mo.:
T. C. Rafferty;

Nominees of the Maccabees, Detroit, Mich.:
W. P. Coler, Roy Deng, Maurice Hartwell, F. H. Lee, J. E. Little;

Nominees of the Massachusetts Institute of Technology:
Professors Charles Edward Fuller, Harry Manley Goodwin, William Hovgaard, Frederick George Keyes, Edward Pearson Warner;

Nominees of the Missouri State Life Insurance Company, St. Louis, Mo.:
Lucy Andrews, O. J. Burian, Alfred Jekel, B. E. Shepherd, Douglas Wood;

Nominees of the National Life Insurance Company of the United States of America, Chicago, Ill.:

Miguel Antonio Basoco, University of Chicago; Esther Comegys, Wellesley College, Chester George Jaeger, University of Missouri; Herbert S. Thurston, Brown University;

Nominees of the Pacific Mutual Life Insurance Company, Los Angeles, Calif.:

Waldo D. Boss, Leslie J. Cooper, Alfred G. Hann, Arthur N. Havens, William H. Otis, Oscar Swenson, Ellis E. Thomas, Cecil V. VanWyck, Roy Wheeler;

Nominees of the Philadelphia Electric Company, Philadelphia, Pa.:

Raymond Bailey, C. W. Bates, P. H. Chase, John L. Conner, N. E. Funk, Howard S. Phelps, Farley R lston, George Schleicher, W. G. Wagner, Robert A. Walton;

Nominees of the Prudential Insurance Company, Newark, N. J.:

R. S. Brush, A. H. Fitzgerald, F. D. Kineke, E. L. Lundgren, G. S. Mower;

Nominees of the Travelers Insurance Company, Hartford, Conn.:

Charles H. Davis, James Strode Elston, B. D. Flynn, Joseph D. Flynn, Edward Bontecou Morris.

The Council announced the appointment of the following committees: Professor W. B. Ford, Mr. Robert Henderson, Professors D. N. Lehmer, E. J. Townsend, and Oswald Veblen (Chairman) to nominate officers and members of the Council for 1926; Professors R. C. Archibald, W. D. Cairns, Arnold Dresden, W. A. Luby, and E. B. Stouffer (Chairman) to arrange for the Western Christmas meeting in Kansas City, December, 1925. Professor Harris Hancock will represent the Society at the semicentennial anniversary of Vanderbilt University, October 15-19, 1925.

The invitations of Hunter College, New York City, for the next Annual Meeting; of the Ohio State University for the Summer Meeting in 1926; and of the University of Wisconsin for the Summer Meeting and a Colloquium in 1927, were accepted with the thanks of the Society.

It was announced that the Bulletin for 1926 is to be printed by the George Banta Publishing Company, Menasha, Wisconsin, and that it has been decided to continue the issue of six numbers annually. The problem connected with the printing and distribution of the Colloquium Lectures was referred to the Committee on Printing.

Professor J. L. Coolidge presented his resignation as chairman of the Committee on Endowment to take effect at the end of the present year. Professor Arnold Dresden also resigned. The goal set by the Committee has been reached, provided the present number of Sustaining Memberships can be maintained. Professor Tomlinson Fort (Chairman) and Dr. R. W. Burgess were added to the Committee.

A report of the Congress on the use of Esperanto in Science was received from Professor Fréchet, the delegate of the Society; an abstract appears in the present issue of this BULLETIN. President Birkhoff announced that he had sent a congratulatory cable to the London Mathematical Society on the occasion of a dinner to celebrate its sixtieth anniversary.

The Council adopted resolutions of thanks to the Assistant Secretary for his able and devoted service in carrying the additional duties of the Secretary during the six months absence of the latter in Europe, and to Mrs. Anna J. Pell for a contribution to the Endowment Fund in honor of her late husband, Professor Alexander Pell.

Titles and abstracts of the papers read before the sectional sessions of the Society follow below. The papers numbered 1 to 22 were read before the section of Geometry on Thursday morning, Vice-President Evans presiding; numbers 23 to 31 before the section of Point Sets and Foundations, Thursday morning, Vice-President Hildebrandt presiding; numbers 32 to 46 before the section of Analysis, Friday morning, President Birkhoff presiding; and numbers 47 to 60 before the section of Algebra and the Theory of Numbers, Friday morning, Professor Bell presiding. Mr. Agnew was introduced by Professor McKelvey, and Professor Schouten by Professor Kasner. Professor R. L. Moore's papers were read by Professor Bennett, and the papers of the following authors were read by title: Brinkmann, Dines, Eiesland (second paper), Ettlinger, Ford, Garabedian, Gronwall, Murnaghan, Nelson, Rainich, Schouten, Tamarkin and Wilder, Vandiver, Weisner (second paper), Wilson.

1. Professor F. R. Sharpe: *Space involutions having a web of invariant rational surfaces.*

The author shows that if the web has at least one simple basis point the involution can be mapped on a rational cubic variety. In all the cases found the cubic variety has at least one double point and can therefore be mapped on a second space. The involution is therefore reducible to a perspective monoidal involution. If there is no simple basis point there is a finite number of independent types of involutions.

2. Professor W. B. Carver: *Note on six points in a plane and the six conics determined by them.*

The author notes certain properties of a set of six points in a real projective plane which are invariant under the group of real linear transformations, the six points not lying on a conic and no three of them being collinear. Two characteristics of each point with respect to the set may be noted: (1) Each point P_i is either inside or outside of the conic C_i determined by the remaining five points; or, briefly, each point is an *inside* or *outside* point of the set. (2) Through each point P_i there pass five conics C_j, $j \neq i$, with tangents t_j at P_i; and also there are five lines l_j joining P_i to each of the other five points P_j; the five tangents t_j are projective with the five lines l_j; and hence, going round the point P_i, the two sets of lines t_j and l_j are met either in the same or in opposite order; and one may say that P_i is respectively a *consistent* or a *contradictory* point of the set. Every set of six points is of one of three types: (A) Each of the six points is inside and consistent. (B) Two points are inside and consistent, and four are outside and contradictory. (C) Three points are inside and contradictory, and three are outside and consistent. If the six points are "nearly" on a conic, the set is of type (C).

3. Professor T. R. Hollcroft: *On the reality of singularities of plane curves.*

By the use of Plücker's equations, Klein's theorem, and the Lefschetz postulate, the maximum number of real cusps of a curve of given order and genus is found. Limits for other real and imaginary singularities are also derived. The maximum number of cusps of a curve of given order and genus has been found by Lefschetz. It is known that, in general, not all of these cusps can be real, and limits for the number of real cusps are found in each case.

4. Dr. Louis Weisner: *Self-projective plane 5-points.*

The problem considered in this paper may be stated as follows: Determine the largest collineation group under which an arbitrary set of five coplanar points, no three of which are collinear, is invariant, and determine all special 5-points which are invariant under a larger group. It is shown

that in general a 5-point is invariant only under the identical transformation. There are four special 5-points which are invariant under a larger group. Two of them are imaginary. The two real special 5-points are (1) a set of five points, four of which are the vertices of an isosceles trapezoid whose axis of symmetry passes through the fifth; (2) the vertices of a regular pentagon.

5. Professor A. D. Campbell: *Plane cubic curves in the Galois fields of order* 2^n.

In this paper are completely tabulated, for the first time, the classes of non-degenerate and degenerate plane cubic curves in the finite analytical geometries whose algebras are those of the Galois fields of order 2^n. A typical cubic curve is given for each class. A complete new machinery is set up to test these cubic curves for nodes or cusps and points of inflection, and to discover the cubic curves that have no real points on them. The Hessian of one of these cubics is the curve itself. There occur cubics with 0, 1, 3, 7, or 9 real points of inflection on them. The coefficient k of the xyz term in the equation of a cubic is a remarkable relative invariant. A corresponding net of conics is given for the general cubic (for which net this cubic is the discriminant of a general conic). The polar conics of these cubics have strange properties.

6. Dr. F. H. Murray: *Generalization of certain theorems of Bohl.* Second paper.

As in the first paper, the motion of a dynamical system is assumed to be defined by a system of differential equations in the canonical form, which are satisfied by a set of constant values of the variables; the behavior of the trajectories near this point depends on the nature of the characteristic exponents for the equations of variation. In this paper it is shown that with fewer restrictions on the characteristic exponents, existence theorems can be proved which are similar to those obtained in the first paper; in addition the asymptotic properties of certain trajectories remaining near the fixed point are studied by methods similar to those of Bohl.

7. Professor C. L. E. Moore: *Ricci notation for geometrical products.*

The author shows how to express the usual geometrical products by the use of Ricci's "system ϵ."

8. Mr. G. Y. Rainich: *Projection of a fixed vector on a surface.*

On a surface in ordinary space a vector field is formed by projecting a fixed vector on the tangent plane at each point; this field satisfies a second order differential equation. It can also be given by means of a

scalar field whose values are lengths of projections of the same fixed vector on the normals. The scalar field also satisfies a second order equation. In the case of a minimal surface this differential equation expresses an internal property. On a one-sided minimal surface the solutions are two-valued, a solution changing its sign when we pass from one side to the other. Some of the results are extended to hypersurfaces and curved spaces.

9. Mr. G. Y. Rainich: *Mass in curved space time.*

Contrary to the opinion which seems to be generally accepted, the mass is not connected with the *contracted* but is implicitly contained in the complete (not contracted) Riemann tensor. In the case of the centrosymmetric solution, mass cannot be separated from the electric charge, i. e., it is impossible to separate linearly from the Riemann tensor a part which contains the constant corresponding to the mass and does not contain the electric charge constant, unless we separate time and space. In the spaces obtained as a result of such a separation, mass appears as a residue of a field determined by the Riemann tensor of space-time.

10. Professor B. H. Camp: *Mutually consistent regression surfaces for three-dimensional frequency solids.*

In the normal solid of frequency, all regressions are linear in form, but for solids that admit of non-linear regression there are interrelations between regression surfaces and total regression curves which severely restrict one's freedom in choosing forms to fit observed surfaces. Some mutually consistent sets are described. It becomes apparent that simple polynomial regression surfaces may imply very complicated forms for the total regressions, save in the all linear case only. This fact has applications in various fields. When complicated two-variate regression is encountered, this does not necessarily mean that the functional relationship would not be simple after a third variate was added. It is suggested that the form "y (polynomial in x) equals polynomial in x" is of frequent occurrence among two-way regressions and should be tried instead of "y equals polynomial in x" when the regression fails to be linear.

11. Professor A. A. Bennett: *New properties of an orthocentric system of triangles.*

Let T_1, T_2, T_3 be any triangle, and let T_0 be the orthocenter. Let T_0', T_1', T_2', T_3' be the reflexions of T_0, T_1, T_2, T_3, respectively, in N, the center of the nine-point circle. The author states the following as the result of computation with concomitants of simultaneous quadric and cubic binary forms. There is a pencil of cubic curves through the nine points N, T_0, T_1, T_2, T_3, T_0', T_1', T_2', T_3'. One of these, K_M, passes also through the six midpoints M_1, M_2, M_3, M_1', M_2', M_3', each of which is

the midpoint of two mutually perpendicular sides. Another, K_P, passes through the six pedal points. A third, K_C, is a circular cubic. The real asymptote of K_C is the inflectional tangent of K_M at N and is the second polar with respect to the two circular points of the medial line triple NM_1, NM_2, NM_3. The entire system is determined completely by any one of the eight triangles $T_0T_1T_2$, $T_1T_2T_3$, $T_2T_3T_0$, $T_3T_0T_1$, $T_0'T_1'T_2'$, $T_1'T_2'T_3'$, $T_2'T_3'T_0'$, $T_3'T_0'T_1'$, in contrast to the more familiar figures associated with a triangle.

12. Dr. H. W. Brinkmann: *Solutions of the Einstein equations for empty space.*

Various solutions of the Einstein equations for empty space, $R_{ij}=\frac{1}{4}Rg_{ij}$, are obtained when certain simplifying assumptions are made concerning the coefficients of the fundamental quadratic differential form. In particular, it is shown that the Schwarzschild solar field line element furnishes essentially the only solution for which said quadratic form looks like this: $ds^2=Edx_1^2+2Fdx_1dx_2+Gdx_2^2+A(edx_3^2+2fdx_3dx_4+gdx_4^2)$, where E, F, G, A are functions of x_1, x_2 alone and e, f, g are functions of x_3, x_4 alone.

13. Dr. H. W. Brinkmann: *Einsteinian 4-spaces imbedded in euclidean 5-space.*

In a recent paper (MATHEMATISCHE ANNALEN, vol. 94 (1925), Theorem IX, on p. 140) the author has given an example of a 4-space satisfying the Einstein equations $R_{ij}=0$ (R_{ij} being the contracted Riemann-Christoffel tensor) which may be imbedded in euclidean 5-space. In the present paper it is shown that any 4-space having this property can be put in the form there given.

14. Dr. H. W. Brinkmann: *Riemann spaces of class one.*

A Riemann n-space is of class one if it can be imbedded in euclidean $(n+1)$-space. The present note gives a simple condition which is sufficient to ensure that a given n-space $(n>3)$ be of class one. This condition is not necessary, but is satisfied by such spaces "in general."

15. Dr. H. W. Brinkmann: *The torsion of a Riemann n-space imbedded in a euclidean m-space $(m\geqq n+2)$.*

The torsion of a Riemann space imbedded in a euclidean space is defined and its properties are investigated. It is found necessary to introduce "torsion vectors" and antisymmetric "torsion tensors" of rank two. The most pleasing results are obtained for the case $m=n+2$, where the study of the torsion depends upon that of a single Pfaffian expression.

16. Professor C. A. Garabedian: *Solution of the problem of the thick rectangular plate with two opposite edges supported and two edges free, and under uniform or central load.*

In a recent note on beams (COMPTES RENDUS, vol. 179 (1924), p. 381) space prevented the author from giving more than the central deflection of the plate here considered (the load being uniform). Although the complete solution had been obtained at that time, it was in rather lengthy form. It is interesting to try to express this solution in terms of the thin plate solution and its derivatives, as has been shown possible, for example, in the case of the clamped or supported plate (COMPTES RENDUS, vol. 180 (1925), p. 257). The author has just succeeded in writing the thick plate problem in the concise form desired. And, as in the example cited, the formulas for central load are obtained at once from those for uniform load by merely using the appropriate thin plate solution and suppressing the terms in which the letter p (load) appears explicitly. An abstract of these results will be found in the COMPTES RENDUS. The author plans to give the details in a subsequent paper on beams.

17. Professor C. A. Garabedian: *Rectangular plates of constant or variable thickness.*

This paper is a sequel to the author's *Circular plates of constant or variable thickness* (TRANSACTIONS OF THIS SOCIETY, vol. 25 (1923), pp. 343-398). It develops in detail the method of series in rectangular coördinates, and obtains *usable* solutions of a number of outstanding problems in thick rectangular plates and in beams of rectangular section. The differential equations that now present themselves for solution are *partial* rather than *total*, but happily this increase in complexity is offset by a widening of the field of applications. In circular plates not all the solutions in constant thickness were new, and the differential equations in variable thickness were integrable in certain cases; in the present paper the partial differential equations in variable thickness are somewhat complicated, but the applications to problems in constant thickness are all believed to be new, a situation apparently due to the lack hitherto of a suitable method of attack.

18. Professor J. A. Eiesland: *A generalization of the tetrahedral complex in odd S_{n-1}.* Preliminary report.

If we consider two pencils of flats (P, E), (P', E') in $(n-1)$-space and let a projective relation be established between them, all the flats that intersect corresponding flats in the two pencils will define a quadratic flat-complex analogous to the tetrahedral complex in 3-space. The singular surface of the complex is then considered. It is generated by ∞^2 flats which belong to a linear flat-complex of rank $n-2$. The surface is of $(n+2)/2$ dimensions and has two conics for double edges. This class of spreads has been studied by the author in an earlier paper (TÔHOKU MATHEMATICAL

JOURNAL, vol. 16). Its transform in sphere-space is a generalized Dupin cyclide of type $n-2$. The result translated into sphere-space is as follows: The singular surface of a quadratic sphere-complex, consisting of all the spheres which touch corresponding spheres of two projective pencils of spheres, is a Dupin cyclide of the type $n-2$.

19. Professor J. A. Eiesland: *The loci of point singularities on a generalized Kummer surface in odd S_{n-1}.* Preliminary report.

The author studies the loci of double, triple, and quadruple points on the generalized Kummer surface in flat-geometry; their respective orders are obtained, and an important contact-locus on the surface is also examined.

20. Dr. C. A. Nelson: *Note on rational plane cubics.*

This paper appears in full in the present issue of this BULLETIN.

21. Dr. T. Y. Thomas: *A projective theory of affinely connected manifolds.*

This paper gives a reduction of the projective theory of a given n-dimensional manifold of affine connection to the affine theory of an associated manifold involving one additional dimension. Thus the ordinary methods of the Ricci calculus may be applied immediately to the development of the projective geometry. The associated manifold is subject to a particular group of transformations, but this causes no difficulty and one easily arrives at a method of projective extension, a system of projective normal coördinates, the replacement (reduction) theorem, a complete set of projective invariants, etc., all of which have their counterpart in the ordinary affine theory of the manifold. The paper will be published in the MATHEMATISCHE ZEITSCHRIFT.

22. Professor F. D. Murnaghan: *A study of the conformal mapping $w=az+b/z+c/z^2$, and its application to aerodynamics.*

The mapping $w=az+b/z+c/z^2$ sends the unit circle in the z plane into a bicircular quartic and the constants may be chosen so that the curve has a cusp and has the general shape of the well known Joukovsky profiles. The left and turning moment are calculated in the usual way.

23. Professor A. D. Campbell: *On the use of fractions in the algebra of logic.*

In this paper a proposition such as "All a is b," when used as a premise or conclusion of a syllogism, is written algebraically as a fraction a/b.

Four rules are given for the use of these fractions in the algebraic representation of syllogisms. For example, the syllogism "All b is c, all a is b, therefore all a is c" is expressed algebraically by writing $(b/c)(a/b)<a/c$. All the types of syllogisms are then given algebraically, using these fractions.

24. Professor R. L. Moore: *Concerning the relation between separability and the proposition that every uncountable point set has a limit point.*

In this paper it is shown that in order that every subset of a given class D of Fréchet should be separable it is necessary and sufficient that every uncountable subclass of that class should *have* a limit point, and in order that this condition should be fulfilled it is necessary and sufficient that every uncountable subclass of D should *contain* a point of condensation of itself. It follows that a class D is separable if, and only if, every subset of it is separable.

25. Professor R. L. Moore: *Concerning the separation of point sets by curves.*

The following proposition is proved and, with its help, the author establishes other results concerning accessibility and the separation of point sets by simple closed and open curves. Theorem: If T, the common part of two bounded continua M and K, is totally disconnected and neither M nor K separates the plane, and $K-T$ is connected, then there exists a simple closed curve which encloses $K-T$ but no point of M and which contains T but no point of $M+K-T$.

26. Dr. R. G. Lubben: *The double elliptic case of the Lie-Riemann-Helmholtz-Hilbert problem of the foundations of geometry.*

Hilbert has formulated a set of axioms concerning a group of motions which is sufficient to necessitate that this group be simply isomorphic with either the euclidean or the Bolyai-Lobatschefskian group of rigid motions in the plane. He assumes, however, that the set of points which undergoes the transformation is a number manifold. R. L. Moore has given a treatment in which this assumption is not made in advance, but in which there is a simultaneous analysis of the group of transformations and of the space which undergoes this transformation. In the present paper we shall give a similar analysis for the double-elliptic case. The problem (solved by Hilbert for the euclidean case by methods which do not apply here) of proving the congruence of the base angles of an isosceles triangle takes up a large part of the paper and presents particular difficulties. The problem is solved with the use of trigonometry built up by methods similar, in part, to those used by W. H. Young for the euclidean and Bolyai-Lobatschefskian spaces.

27. Dr. R. G. Lubben: *Concerning limiting sets.*

Let G be a collection of point sets and let K be the accumulation set of G. The following results are established in this paper. (1) If K is bounded and P is a point of K, there exists a sub-collection of G which has a sequential limiting set containing P. (2) There exist at most a countable number of elements of G which are not subsets of K. (3) If the elements of G are connected point sets, and M is a maximal connected subset of K, then there exists a sub-collection of G whose accumulation set is M. (4) If G is a collection of connected point sets, then a necessary and sufficient condition that K be connected is that there exist a sub-continuum M of K such that the accumulation set of every infinite sub-collection of G contains points of M. (5) If G is a bounded collection of closed point sets, then, if for each positive number e and each point P belonging to an element of G there exist an element g_{eP} of G whose upper distance from P is less than e, G is a separable set of sets.

28. Dr. R. G. Lubben: *Surrounding theorems with applications to questions of accessibility.*

The author gives the following generalization of surrounding theorems established by Zoretti and by R. L. Moore. If K is a bounded point set and if L is either K or a closed point set consisting of the points belonging to a collection of maximal connected subsets of K, and e is a positive number, then there exist a finite number of domains whose sum covers L such that no one of these domains plus its boundary contains any points or boundary points of any of the remaining domains; no point or boundary point of one of these domains is at a distance from L greater than e; the boundary of each domain consists of a finite number of simple closed curves none of which contain a point of K. The author also shows that if M and N are bounded point sets, M' and N' have no points in common, and there exists an arc AXB which contains no point of M and an arc AYB which contains no point of N, then there exists an arc AZB which contains no point of $M+N$. The author also states a necessary and sufficient condition that a boundary point of domains be accessible from the domain.

29. Dr. L. M. Graves: *Taylor's theorem in general analysis.*

Consider an abstract set X of elements x, which together with certain operations of addition, multiplication by real or complex numbers a, and taking the modulus, constitute a space called by Fréchet "*espace (D) vectoriel*" (Fréchet, COMPTES RENDUS, vol. 180 (1925), p. 419; Banach, FUNDAMENTA MATHEMATICAE, vol. 3 (1922), p. 134). Let Y be a second such space, which is also complete, in the sense that every regular sequence has a limit in the space. Let F be a function defined and of the class $C^{(n)}$ on a convex region X_0 of X, and with functional values in the space Y. The definition of the class $C^{(n)}$ generalizes that given by Bolza in his *Variationsrechnung*, page 14. Then the function F may be expanded about

an arbitrary element x of the region X_θ in a Taylor's series with remainder. The formula for the remainder generalizes one given by Jordan in his *Cours d'Analyse* (vol. 1, p. 247 (2d edition), formula (2)). A fuller account of the results obtained is found in a note by the author in the COMPTES RENDUS, vol. 180 (1925), p. 1719.

30. Professor W. A. Wilson: *On the oscillation of a continuum.*

If c is a point of a continuum A and C_δ is a sub-continuum of A which contains all points of A whose distance from c is less than δ, then the lower bound of the diameters of the sub-continua C_δ for all positive values of δ is a function $t(c)$. This function has properties similar to those of the oscillatory function $\sigma(c)$ of Mazurkiewicz and might well be used as a definition of oscillation. Closely associated with this function are certain sub-continua of A called *oscillatory sets*. These concepts have proved to be useful tools in the study of continua, particularly irreducible continua. In this paper some of their properties for continua in general are given, and they are used to investigate the properties of the oscillation of a continuum in the vicinity of points where it differs from zero.

31. Professor W. A. Wilson: *Some properties of a continuum limited and irreducible between two points.*

For a limited continuum ab irreducible between two points the oscillatory set about each point is unique and has a diameter equal to the oscillation of the continuum at that point. Oscillatory sets of a certain type are called complete; each of these is a continuum of condensation and defines a partition of ab into two continua having this set as a common part. When the points of the *first genre* are everywhere dense, it is found that each point of ab lies on one and only one complete oscillatory set and that the aggregate of these sets has the order type of the linear continuum. From this it follows that there is a correspondence $x=f(t)$ between the points $\{x\}$ of ab and $\{t\}$ of a linear segment cd such that $f(t)$ is one-valued and continuous at an everywhere dense set of points and, at the remaining points $\{t'\}$ of cd, $f(t)$ is many-valued and has as its values the aggregate of limiting values of $f(t)$ as t approaches t' over all possible sequences.

32. Professors E. R. Hedrick and M. B. Porter: *A proof of Weierstrass's theorem with application to Dirichlet's principle.*

In this paper, a new method of demonstrating Weierstrass's theorem is given, which is capable of immediate generalization to functions of several variables, and which is fundamentally simpler in nature than former proofs, particularly for functions of several variables. Applications to the proof of Dirichlet's principle and to the proof of the existence of Green's functions are indicated.

33. Professors E. R. Hedrick and M. B. Porter: *On a generalization of Gibbs' phenomenon.*

The phenomenon usually called *Gibbs' phenomenon*, in the case of Fourier series, is here expressed by means of the point oscillation ω of the limiting function $f(x)$, and the two-dimensional point oscillation Ω of the approximating trigonometric polynomials $\phi_n(x)$, thought of as functions of x and n. The phenomenon as described by Bôcher leads to a ratio Ω/ω, which may be called the *Bôcher ratio*. The concepts involved are generalized in this paper to approximating functions $\phi(x, y)$ to any function $f(x)$; and it is shown that if $f(x)$ is continuous and if $\Omega=\omega$, the approach is uniform, and conversely, so that for a continuous limit, the equation $\Omega=\omega$ is both necessary and sufficient to insure uniform approach. Generalizations to the case when $f(x)$ is discontinuous are immediate, and are stated. The applications of these theorems to the discussion of uniform approach are discussed. In particular, it is pointed out that any Dirichlet function can be approached by trigonometric polynomials either without a Gibbs' effect at any point, or else with preassigned Gibbs' effects at each of a preassigned finite number of points.

34. Professor R. G. D. Richardson: *A problem in the calculus of variations with an infinite number of auxiliary conditions.*

In ordinary problems of the calculus of variations involving one or more auxiliary conditions, an integral possesses either a maximum or a minimum, the other extremum being infinite. By imposing an infinity of auxiliary conditions, we may discuss problems where both maximum and minimum exist. In the simplest problem treated, $\int_0^1 (pu'^2 - qu^2)dx =$ extremum, $\int_0^1 ku^2dx = 1$, $\int_0^1 kU_iudx = 0$, $(i=1, \cdots, m-1;\ s+1, \cdots)$, where U_i are solutions of the differential equation $(pu')'+qu+\lambda ku=0$, $u(0)=u(1)=0$, $p>0$, $k>0$, corresponding to the characteristic numbers λ_i, the Euler and Jacobi conditions, together with the Hamiltonian function and the Hilbert integral, appear as limiting cases of the corresponding formulas for the finite problem. But the corresponding analogues of the Legendre and Weierstrass conditions suggest $p>0$ for a minimum and $p<0$ for a maximum, and this contradiction obviously demands reexamination of the bases of validity of these criteria. A discussion of the Jacobi condition involving an infinite determinant of integrals is vital for the treatment of the problem, and it is found that the Legendre and Weierstrass conditions, when of any significance whatever, must play a role subsidiary to it.

35. Professor C. N. Moore: *On convergence factors in multiple series.*

Convergence factors may be conveniently divided into two classes. The first class includes those which merely have the property of reducing

a divergent series to convergence or increasing the rapidity of convergence of a convergent series throughout a certain range of values of the parameters of which the convergence factors are functions. The second class includes those which have the additional property of assigning a value to the original series by the process of allowing the parameters to approach certain limit-values, this value being consistent with the definition of convergence or one of the standard definitions of summability. Necessary and sufficient conditions for convergence factors of both types in the case of double series summable by Cesàro's method have been obtained in a paper previously presented to the Society (this BULLETIN, vol. 30 (1924), p. 220). In the present paper, these results are generalized to the case of multiple series of any order.

36. Professor Marston Morse: *Relations between the singular points of n ordinary differential equations of the first order.*

Poincaré, in a series of four papers *Sur les courbes définies par les équations différentielles*, has included among his results certain relations between the numbers of the different types of singular points in regions bounded by manifolds "without contact." His relations are limited to a number of special cases in which $n=1$, 2, or 3, and in which the number of bounding manifolds is 1 or 2. In the present paper the relations recently obtained by the author between the different types of critical points of a function of n variables serve as a means of arriving at a general formulation of the relations of which Poincaré's results are special cases.

37. Professor Einar Hille: *A class of reciprocal functions.*

This paper contains a study of integrals of the form

$$h(z)=\frac{1}{\sqrt{\pi}}\int_{-\infty}^{+\infty} e^{-(z-t)^2}g(t)dt$$

with varying assumptions concerning $g(t)$. Reciprocal relations between $h(z)$ and $g(t)$ are noted. Incidentally the paper shows many connections with the theory of Hermite's polynomials and with Fourier's integral theorem, and gives a contribution to the summation theory for Hermite's series. The paper marks a first step in the author's study of integral equations with the kernel $\exp[-(x-y)^2]$ and the range $(-\infty, +\infty)$.

38. Professor J. Tamarkin: *Boundary problems and expansion theorems in the theory of integro-differential equations.*

In this paper the integro-differential equation

$$L(y)=p(x)\int^{b} q(t)M(y)dt+f(x)$$

with boundary conditions of the form

$$A_i(y)\,|_{x=a}+B_i(y)\,|_{x=b}+\int_a^b M_i(y)dt=0, \qquad (i=1, 2, \cdots),$$

is discussed, where $L(y)$ denotes a linear differential operator of order n, and $A_i(y)$, $B_i(y)$, $M_i(y)$, $M(y)$ are analogous operators of orders not greater than $(n-1)$. The coefficients of all these operators are polynomials in a certain complex parameter ρ, whose coefficients are functions of the independent variable x. Various problems concerning the existence and asymptotic expressions of characteristic numbers and fundamental functions of the Green function, as well as the general expansion problem of an arbitrary function in series of fundamental functions are solved. The methods used in the paper are essentially those previously used by Birkhoff and the present author.

39. Professors J. Tamarkin and C. E. Wilder: *Second law of the mean in the theory of definite integrals.*

This paper will appear in full in an early issue of this BULLETIN.

40. Professor J. A. Shohat: *On a general formula in the theory of Tchebycheff's polynomials and its applications.*

In this paper the author derives a general formula which gives the solution of the following problem: For all polynomials $A_n(x)=\sum_{i=0}^{n}g_ix^i$ satisfying the condition $|q(x)A_n(x)|\leq M$ on (a, b), find the upper limit of $|\sum_{i=0}^{n}\alpha_ig_i|$. Here $M>0$ and α_i are arbitrarily given real constants, and $q(x)$ is defined on a certain interval (a, b), finite or infinite. The solution is given by means of Tchebycheff's polynomials. Specifying the α_i and $q(x)$, the author derives certain general minimum properties of Tchebycheff's polynomials, and also some inequalities concerning polynomials in general, on a finite or infinite interval. Some of the results thus obtained generalize those given by Tchebycheff, V. Markoff, and Szegö.

41. Professor H. J. Ettlinger: *Note on a fundamental theorem concerning the limit of a sum.*

This paper appears in full in the present issue of this BULLETIN.

42. Professor J. A. Schouten: *On the conditions of integrability of covariant differential equations.*

The problem of finding the conditions of integrability for a system of differential equations has been solved in the general case before. But the direct application of the known results to *covariant* equations does not lead to convenient expressions. In the present paper the author treats

with the aid of intrinsic operations the case where the first members of the equations are linear in the covariant derivatives and have constant coefficients, and the general solution depends on a *finite* number of arbitrary constants.

43. Professor L. R. Ford: *The fundamental region for a Fuchsian group.*

This paper appeared in full in the November-December (1925) issue of this BULLETIN.

44. Mr. R. P. Agnew: *On the form of the solid of revolution of minimum resistance when the normal resistance varies as the nth power ($n>0$) of the normal velocity.*

In this paper the extremals are investigated by means of the Euler, Legendre, and Weierstrass conditions, and solutions are obtained which satisfy the Weierstrass-Erdmann corner condition. The Euler differential equation is integrated when $n=1$, $3/2$, 2, and the loci plotted. Numerical tables are given.

45. Professor O. D. Kellogg: *Some properties of bounded polynomials in several variables.*

Let $P_n(x, y)$ denote a polynomial, of degree not greater than n, with real or imaginary coefficients, the real variables x and y being restricted to the region $x^2+y^2 \leqq 1$. Let $|P_n(x, y)| \leqq 1$ in this region. Then if $P_n(x, y)$ is homogeneous, its coefficients have moduli which do not exceed the corresponding binomial coefficients; these bounds are attained (except for the cases of the coefficients of x^n and y^n) only when $P_n(x, y)$ is a number of modulus 1 times the real or imaginary part of $(x+iy)^n$. The magnitude of the gradient of $P_n(x, y)$ does not exceed n for $x^2+y^2 \leqq 1$. Whether $P_n(x, y)$ is homogeneous or not its gradient cannot exceed n^2 for $x^2+y^2 \leqq 1$. These results are proved, and generalized to polynomials in m variables, by elementary considerations based on Descartes' rule of signs. Connection is made with the theorems of Bernstein and Markhoff.

46. Professor L. L. Smail: *Simplification of a general method of summability.*

This note gives a simplification of the conditions and proofs of a method of summability of divergent series discussed by the author in the ANNALS OF MATHEMATICS ((2), vol. 20, pp. 149-154).

47. Professor G. A. Miller: *Multiply transitive substitution groups.*

A substitution group G of degree n is said to be r-fold transitive if each of the $n(n-1)\cdots(n-r+1)$ permutations of its n letters taken r at a time is represented by at least one substitution of G. It is not sufficient to say that each of its possible sets of r letters is transformed into every other one of these sets by substitutions of G. Another well known definition of an r-fold transitive group of degree n is that this group contains a transitive subgroup of each of the degrees $n, n-1, \cdots, n-r+1$. These two definitions are equivalent and if a group is r-fold transitive it is also $(r-\alpha)$-fold transitive, where α is a positive integer $\leqq r-1$. Although a study of multiply transitive groups of degree n which do not involve the alternating group of this degree was successfully undertaken by E. Mathieu, who found one such 5-fold transitive group of each of the degrees 12 and 24, little progress along this line has been made since then. The main theorem of the present article may be stated as follows: *If a substitution group is of degree $lp+k$, p being a prime number, $p>l<k$, then this group cannot be more than k-fold transitive, $k>2$, unless it is the alternating or the symmetric group.*

48. Professor O. E. Glenn: *A program on ordinary differential parameters.* Preliminary report.

The method of typical representation which the author has developed in a previous paper furnishes, with great generality, explicit complete systems of differential parameters of binary forms. The present paper extends this subject and, when completed, is to give also certain lists of differential combinants and parameters derived from them by translation, as explained by the author in a recent article in the PROCEEDINGS OF THE OF THE NATIONAL ACADEMY (vol. 11, No. 6).

49. Dr. Louis Weisner: *Groups in which the normaliser of every element except identity is abelian.*

This paper has appeared in full in the October (1925) number of this BULLETIN.

50. Professor W. L. G. Williams: *On the formal modular invariants of binary forms.*

This paper owes its origin to an attempt to determine a fundamental system of formal modular invariants, mod p, a prime, of the binary cubic form. Several general methods of finding formal modular invariants are developed, and the discussion of two special cases makes clear a fundamental difference between the cases where p is of the form $3m+1$ and of the form $3m+2$.

51. Mr. H. S. Vandiver: *Application of the theory of relative cyclic fields to both cases of Fermat's last theorem.*

In this paper the author attacks the Last Theorem by a new method based on the theory of power characters in the field $\Omega(\theta)$, where θ is a primitive (pk)th root of unity, k prime to the odd prime p. The following result, among others, is obtained: If $x^p+y^p+z^p=0$ is satisfied in integers none zero and all prime to the odd prime p, v is any number in the set t, $1-t$, $1/t$, $1/(1-t)$, $(t-1)/t$, $t/(t-1)$, and $-x/y=t$, then if $\alpha=e^{2i\pi/p}$, $\beta=e^{2i\pi/(n-1)}$,

$$q(n)\prod_{a=1}^{n-2}\left((1-v)\mathrm{ind}(\alpha\beta^a-1)-q(n)\right)\equiv 0 \pmod{p},$$

where $\mathfrak{q}=(\beta-r, n)$, r is a primitive root of n, $(\alpha\beta^a-1)^{(N(q)-1)/p}\equiv\alpha^i(\mathrm{mod}\mathfrak{q})$, $i=\mathrm{ind}(\alpha\beta^a-1)$, $q(n)=(n^{p-1}-1)/p$, n is any prime $\not\equiv 0$ or $1 \pmod{p}$, and $N(\mathfrak{q})$ is the norm of $\mathfrak{q}$.

52. Mr. H. S. Vandiver: *On algorisms for the solution of the quadratic congruence.*

The problem considered here is the setting up of an algorism for the determination of the integer x in $x^2\equiv q \pmod{p}$ where p and q are odd primes, without the use of methods of trial. A scheme is described which depends on the expansion of $\sqrt{pq}$ as a continued fraction. For the case where p and q are each of the form $4n+3$ the method is devoid of trial and includes a proof of the law of quadratic reciprocity. For other forms of p and q the element of trial is introduced.

53. Mr. H. S. Vandiver: *Laws of reciprocity and the first case of Fermat's last theorem.*

This paper contains the proof of the following theorem: Suppose $x^p+y^p+z^p=0$ is satisfied in integers none zero and all prime to the odd prime p. Also, let the principal ideal $(\omega(\alpha))$ be the pth power of any ideal in the field defined by $\alpha=e^{2i\pi/p}$ which is prime to (z) and (p); then

$$f_{p-n}(t)\left[\frac{d^n \log\omega(e^v)}{dv^n}\right]_{v=0}\equiv 0 \pmod{p};$$

$f_k(t)=\sum_{s=1}^{p-1}s^{k-1}t^s$; $t\equiv -x/y \pmod{p}$; $n=1, 2, \cdots, p-2$, and e is the Napierian base. A number of corollaries are obtained to this theorem, including the criteria $f_{p-n}(t)f_n(1-t)\equiv 0 \pmod{p}$, $n=1, 2, \cdots, p-1$.

54. Mr. H. S. Vandiver: *A new theory of the representation of integers as definite quadratic forms.*

This paper presents first a proof of the following theorem: If m is any positive integer, and a is a positive integer prime to m, then there is

at least one and not more than two sets (x, y) such that $ay \equiv \pm x \pmod{m}$, where x and y are integers $0 < x < \sqrt{m}$, $0 < y < \sqrt{m}$, and two sets (x_1, y_1) and (x_2, y_2) are regarded as the same if and only if $x_1y_2 = x_2y_1$. It is also possible, by the use of a certain continued fraction algorism, to determine each set directly. This proof and algorism were given by the author in an earlier paper for the case m a prime. The theorem is shown to furnish a method for finding all the representations of a number N in the form $ax^2+bxy+cy^2$, where $b^2-4ac<0$.

55. Mr. H. S. Vandiver: *Note on the condition that a cubic equation have an integral root.*

This note is devoted to the proof of the following theorem: If $x^3+px+q=0$ and p and q are rational integers, then a necessary and sufficient condition that x be a rational integer is that $(-27q+3\sqrt{-3\Delta})/2$ be the cube of an integer in the algebraic field defined by $\sqrt{-3\Delta}$, where Δ is the discriminant of the given equation.

56. Professor L. L. Dines: *Definite linear dependence.*

In this paper, which will appear in the ANNALS OF MATHEMATICS, the author defines definite linear dependence as follows. The m sets of n real constants (1) $a_{i1}, a_{i2}, \cdots, a_{in}$, $(i=1, 2, \cdots, m)$ are said to be *definitely linearly dependent* if there exist m real constants $c_1, c_2, \cdots, c_m$, of which none is negative and at least one is positive, such that $c_1a_{1j} + c_2a_{2j} + \cdots + c_ma_{mj} = 0$, $(j=1, 2, \cdots, n)$. In a recent paper (ANNALS OF MATHEMATICS, (2), vol. 23 (1922)) Carver found this particular type of linear dependence to be a necessary and sufficient condition for the inconsistency of the system of linear inequalities (2) $a_{i1}x_1+a_{i2}x_2+\cdots+a_{in}x_n>0$, $(i=1, 2, \cdots, m)$. In an earlier paper (ANNALS OF MATHEMATICS, (2), vol. 20 (1919)) the present writer studied this system of inequalities, characterizing the solutions by means of a concept which he called the I-rank of the matrix of elements a_{ij}. In the present paper he establishes the equivalence of the following three conditions: (a) definite linear dependence of the system (1); (b) inconsistency of the system (2); (c) zero I-rank for the matrix of elements a_{ij}. The possibility of extension to the field of continuous functions is noted, and the author hopes to present results along these lines at a later date.

57. Professor L. L. Dines: *On certain symmetric sums of determinants.*

In his second paper, the author presents formal identities of two types. The determinants occurring in the first type are those obtainable from any given determinant by a certain well defined transformation. The determinants occurring in the second type have as elements the elements of a given persymmetric array, affected by binomial coefficients according to a definite law. The paper will appear in the AMERICAN JOURNAL.

58. Professor A. A. Bennett: *Proof that large primes have four consecutive quadratic residues.*

The author proves by elementary methods that for e ch prime $p>53$, there are at least four consecutive positive (non-zero) integers which are reduced quadratic residues, modulo p. A complete set of mutually exclusive possibilities are examined for each of which an actual numerical example of four consecutive quadratic residues is furnished. Certain systems related respectively to the primes 11, 13, 53 require special consideration. One is inclined to venture the conjecture that for each natural number n there exists an N such that for each prime $p>N$ there are at least n consecutive quadratic residues modulo p.

59. Dr. T. H. Gronwall: *The algebraic structure of the formulas in plane trigonometry.* Second paper.

In continuation of his first paper on the subject, read May 2, 1925, the author investigates the formulas connecting the angles of a plane triangle from the point of view of Hilbert's theory of algebraic forms (MATHEMATISCHE ANNALEN, vol. 36 and 42).

60. Professor E. T. Bell: *An algebra of sequences of functions.*

The algebra is devised to facilitate the derivation of relations between the elements of any sequence of functions and those of any given sequences. It is of use in the applications to arithmetic of several types of polynomials, including those given as coefficients (functions of the moduli) in the power series expansions of multiply periodic functions. This paper will appear in the TRANSACTIONS OF THIS SOCIETY.

ARNOLD DRESDEN,
Assistant Secretary.
R. G. D. RICHARDSON,
Secretary.

THE SECOND ITHACA COLLOQUIUM*

The tenth colloquium of the American Mathematical Society and the second Ithaca Colloquium was held at Ithaca, N.Y., in conjunction with the thirty-first summer meeting of the Society on September 8-12, 1925.

Tied as we are to the decimal system of notation, the attainment of a tenth in any sequence is in the nature of the passing of a milestone. So it might be interesting to give in this connection a brief summary of the colloquia which have been held by the Society up to the present time.†

The colloquium idea was an outgrowth of the desire on the part of members of the Society to hear more extensive treatments of some of the recent developments of the science than can be given in the brief papers which are read at regular meetings of the Society. It was originally planned to hold these colloquia annually, but experience seems to have shown that a larger interval was better. They have been held at intervals of from two to five years, but the plan is to hold them at intervals of not more than two years in the future.

The main facts concerning the first ten colloquia are as follows.‡

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* A report prepared by Professor T. H. Hildebrandt at the request of the Secretary of the Society and the editors of this BULLETIN.

† A résumé of the first five colloquia is to be found in the report of V. Snyder on *The Fifth Colloquium*, this BULLETIN, vol. 13 (1906-7) pp. 72-73. Material covering the first seven is contained in the preface to the volume of *Madison Colloquium Lectures*. See also the list of colloquium lectures published by this Society, on the inside of the front cover of this number.

‡ Perhaps the Evanston Colloquium, held on August 28 to September 9, 1893, should count as the *zeroth* or as the *preliminary* colloquium. Although it was not held under the auspices of this Society (which was not yet existent as such), the Society did acknowledge its interest and indebtedness by republishing the lectures in 1911 (reported by A. Ziwet and first published in 1893).

THE FIRST COLLOQUIUM. Buffalo, N.Y., September 2-5, 1896. Attendance, 13.

M. BÔCHER: *Linear Differential Equations and their Applications.* Not published in full, but parts are contained in a pamphlet on *Linear Differential Equations* (Harvard University, 1898), and a paper in the ANNALS OF MATHEMATICS, (1), vol. 12.

JAMES PIERPONT: *Galois Theory of Equations.* Published in the ANNALS OF MATHEMATICS, (2), vols. 1 and 2.

THE SECOND COLLOQUIUM. Cambridge, Mass., August 22-27, 1898. Attendance, 26.

W. F. OSGOOD: *Some Methods and Problems of the General Theory of Functions.* Published in this BULLETIN, vol. 5, pp. 59-87.

A. G. WEBSTER: *The Partial Differential Equations connected with Wave Propagation.* Not published.

THE THIRD COLLOQUIUM. Ithaca, N.Y., August 21-24, 1901. Attendance, 25.

O. BOLZA: *The Simplest Type of Problems in the Calculus of Variations.* Expanded to his book on *Calculus of Variations* in the DECENNIAL PUBLICATIONS OF THE UNIVERSITY OF CHICAGO.

E. W. BROWN: *Modern Methods of Treating Dynamical Problems and in Particular the Problem of Three Bodies.* An abstract was published in this BULLETIN, vol. 8 (1901-2), pp. 103-113.

THE FOURTH COLLOQUIUM. Boston, Mass., September 2-5, 1903. Attendance, 31.

E. B. VAN VLECK: *Selected Topics in the Theory of Divergent Series and Continued Fractions.*

H. S. WHITE: *Linear Systems of Curves on Algebraic Surfaces.*

F. S. WOODS: *Connectivity of Euclidean Spaces.*

Published for the Society under the title, *The Boston Colloquium*, by the Macmillan Company, in 1905.

THE FIFTH COLLOQUIUM. New Haven, Conn., September 5-8, 1906. Attendance, 43.

E. H. MOORE: *Theory of Bilinear Functional Operations.*

E. J. WILCZYNSKI: *Projective Differential Geometry.*

MAX MASON: *Selected Topics in the Theory of Boundary-Value Problems of Differential Equations.*

Published for the Society under the title, *The New Haven Colloquium*, by the Yale University Press, in 1910.

THE SIXTH COLLOQUIUM. Princeton, N. J., September 15-17, 1909. Attendance, 28.

G. A. BLISS: *Fundamental Existence Theorems.*

E. KASNER: *Geometric Aspects of Dynamics.*

Published by the Society under the title, *Princeton Colloquium Lectures*, in 1913.

THE SEVENTH COLLOQUIUM. Madison, Wis., September 10-13, 1913. Attendance, 51.

L. E. DICKSON: *Certain Aspects of a General Theory of Invariants with special Consideration of Modular Invariants and Modular Geometry.*

W. F. OSGOOD: *Selected Topics in the Theory of Analytic Functions of several Complex Variables.*

Published by the Society under the title, *The Madison Colloquium Lectures*, in 1914.

THE EIGHTH COLLOQUIUM. Cambridge, Mass., September 6-8, 1916. Attendance, 69.

G. C. EVANS: *Theory of Functionals.*

O. VEBLEN: *Analysis Situs.*

Published by the Society under the title, *The Cambridge Colloquium Lectures*, in 1918 and 1922.

THE NINTH COLLOQUIUM. Chicago, Ill., September 8-11, 1920. Attendance, 90.

G. D. BIRKHOFF: *Dynamical Systems.*

F. R. MOULTON: *Theory of Functions of Infinitely Many Variables.*

To be published by the Society.

THE TENTH COLLOQUIUM. Ithaca, N.Y., September 8-12, 1925. Attendance, 122.

L. P. EISENHART: *The New Differential Geometry.*

D. JACKSON: *The Theory of Approximation.*

The committee in charge of the present colloquium consisted of L. E. Dickson, H. H. Mitchell, J. H. Tanner, and H. S. White. Professor Dickson resigned before the choice of lecturers, and H. H. Mitchell was appointed chairman of the committee in his stead. Two groups of five lectures each were given in the Lecture Room of Baker Laboratory, one by each lecturer on Tuesday afternoon, Wednesday afternoon, Thursday evening, Friday afternoon, and Saturday morning. The following one hundred twenty-two persons, the largest number in attendance at any colloquium held up to the present time by the Society, were present at the lectures.

C. R. Adams, R. B. Adams, W. E. Anderson, Archibald, Bacon, Bareis, Bell, A. A. Bennett, T. Bennett, H. Betz, Birkhoff, Bliss, H. S. Brown, M. Buchanan, W. G. Bullard, R. W. Burgess, Bussey, E. B. Callahan, B. H. Camp, A. D. Campbell, C. W. Carter, Carver, J. T. Colpitts, Coolidge, L. P. Copeland, Court, E. F. Cox, C. F. Craig, Denton, Dresden, Dunkel, Eiesland, Eisenhart, G. C. Evans, Everett, F. Farnum, W. B. Ford, Fort, Fry, Gaba, Gale, D. C. Gillespie, Gilman, M. C. Graustein, W. C. Graustein, L. M. Graves, E. R. Hedrick, Herr, Hildebrandt, Hille, Hollcroft, A. M. Howe, Hurwitz, Ingraham, D. Jackson, O. D. Kellogg, E. H. Kennard, B. F. Kimball, Kuhn, W. D. Lambert, Lefschetz, C. A. Lindeman, Long, Lubben, J. V. McKelvey, M. M. McKelvey, MacColl, MacCreadie, MacDuffee, Maria, H. A. Merrill, Michal, G. A. Miller, N. Miller, H. H. Mitchell, Molina, C. L. E. Moore, C. N. Moore, D. S. Morse, H. M. Morse, F. H. Murray, Nassau, Olds, Olson, F. W. Owens, L. R. Perkins, Poritsky, Ranum, Rasor, R. G. D. Richardson, D. E. Richmond, H. Rickard, H. L. Rietz, Schelkunoff, Seely, Sellew, H. C. Shaub, Shewhart, Shohat, Slaught, Smail, C. E. Smith, H. F. Smith, Snedecor, Sullivan, J. Tamarkin, Tappan, J. H. Taylor, E. M. Thomas, J. M. Thomas, T. Y. Thomas, B. M. Turner, Vivian, G. W. Walker, W. J. Wallip, Watkeys, H. E. Webb, Wedderburn, Weisner, W. L. G. Williams, Williamson, Yeaton.

Below are the synopses of the lectures as prepared by the speakers, one lecture being devoted to each section, with the exception of the material in Section III of the series by Professor Eisenhart, which was covered in his third and fourth lectures.

THE NEW DIFFERENTIAL GEOMETRY

BY PROFESSOR L. P. EISENHART

I. *Riemannian Geometry.* The fundamental quadratic form and the measurement of lengths and angles. Geodesics. Geodesic coordinates and Riemannian coordinates. Parallelism of Levi-Civita; its intrinsic character. Curvature of a curve. Associate directions of Bianchi. Parallelism and associate directions in a sub-space.

II. *Linear Connection of a Space.* Infinitesimal displacement of a contravariant vector. Coefficients of the linear connection. Covariant differentiation. The generalized identities of Ricci and the generalized curvature tensor. Fundamental tensors of the second order. Finite displacement of a contravariant vector along a curve. Displacement of a contravariant vector around an infinitesimal closed circuit. Infinitesimal displacement of a covariant vector. Schouten's generalization. The paths of the space. Curvature of a curve and the associate direction of a vector displaced along a curve. Linear connections determined by n independent contravariant vectors (Weitzenboeck). Different linear connections for which the directions of a displaced vector are the same. Schouten's semi-symmetric linear connections.

III. *Geometry of Paths.* Affine connection of a space. Normal coordinates (Veblen). Path coordinates. Affine connections with the same paths. Projective geometry of paths. The Weyl tensor and projective flat-spaces. Normal affine connection (Cartan). Projective geometry of a path (T. Y. Thomas). Projective normal coordinates (Veblen and J. M. Thomas). Projective invariants. First integrals of equations of paths. Weyl's affine connection.

IV. *Geometry of a Sub-Space of a Linearly Connected Space.* Components in the sub-space of a tensor in the enveloping space. Induced linear connection of the sub-space. Relative curvature and associate direction of a vector displaced along a curve. Generalized equations of Gauss and Codazzi. Generalizations of lines of curvature and asymptotic lines.

THE THEORY OF APPROXIMATION,
BY PROFESSOR DUNHAM JACKSON

I. *The Approximate Representation of Continuous Functions.* Weierstrass's theorem. Approximate representation by trigonometric sums in the case of a function satisfying the Lipschitz condition. Approximate representation of functions satisfying other conditions of continuity. Corresponding theorems on polynomial approximation. Convergence and degree of convergence of the Fourier series for a continuous function. Convergence and degree of convergence of the Legendre series for a continuous function. Other forms of approximating function.

II. *Discontinuous Functions and Functions of Limited Variation.* Riemann's theorem on the Fourier coefficients of an arbitrary function. Convergence of the Fourier series in an interval of continuity. Convergence at a point of discontinuity. Uniformity of convergence in an interval of continuity. Convergence in the case of a function of limited variation. Order of magnitude of the coefficients under hypotheses of limited variation; order of magnitude of the mean square error. Degree of convergence in an interval of continuity. Partial analogies in the case of Legendre series.

III. *The Principle of Least Squares and its Generalizations.* The least square property of the Fourier series. Bernstein's theorem on the derivative of a trigonometric sum. Proof of convergence on the basis of the least-square property. Convergence under relaxation of the requirement of a minimum. Convergence with a weight function. Convergence with an arbitrary positive power of the error. Bernstein's theorem for polynomials. Polynomial approximation: Legendre series, series of Tchebycheff polynomials, and approximation with an arbitrary power of the error; convergence throughout the interval; convergence in the interior of the interval. Polynomial approximation over an infinite interval. Other generalizations.

IV. *Interpolation.* Formal solution of the problem of trigonometric interpolation with an odd or even number of points equally spaced over a period. Formal solution of the problem with unequally spaced points. Convergence and degree of convergence of the interpolation formulas for a continuous function, in the case of equally spaced points. An interpolation formula analogous to Fejér's integral in the theory of Fourier series. Convergence and degree of convergence of the interpolation formulas for a discontinuous function. A special problem of polynomial interpolation.

V. *The Geometry of Function Space.* Geometrical significance of the principle of least squares; distance and angle in function space. Application to the theory of statistics. The geometry of frequency functions. Theorems of orthogonality in space of many dimensions. Beginnings of vector analysis.

As announced elsewhere in the present issue of this BULLETIN, the publication of these colloquium lectures has been referred by the Council of the Society to the Committee on Printing. It is anticipated that they will be published in full at an early date.

T. H. HILDEBRANDT

THE OCTOBER MEETING IN NEW YORK

The two hundred forty-fourth regular meeting of the Society was held at Columbia University, on Saturday, October 31, 1925, extending through the usual morning and afternoon sessions. The attendance included the following fifty members of the Society:

Archibald, W. L. Ayres, Babb, C. R. Ballantine, J. P. Ballantine, A. A. Bennett, R. W. Burgess, Alonzo Church, Cole, Fite, Fort, Philip Franklin, Frink, Gronwall, C. C. Grove, Hille, Dunham Jackson, Joffe, Kline, Langman, Lefschetz, Lieber, Longley, H. P. Manning, Maria, Meder, Rainich, Raynor, Reddick, R. G. D. Richardson, Ritt, Schelkunoff, Seely, Siceloff, Simons, Sosnow, M. H. Stone, Stouffer, T. Y. Thomas, Tracey, Tyler, Vallarta, Veblen, Weiss, Anna Pell Wheeler, H. S. White, Whittemore, Wiener, K. P. Williams, W. A. Wilson.

The following nine persons were elected to membership in the Society:

Mr. Ralph Palmer Agnew, Cornell University;
Mr. Julius Alsberg, consulting engineer, New York City;
Dr. Theodore Bennett, University of Illinois;
Dr. Elbert Frank Cox, West Virginia Collegiate Institute;
Mr. M. E. Hekimian, Larchmont, N.Y.;
Mr. Charles Everett Rhodes, Heidelberg University;
Mr. Homer B. Snook, Shaker Heights Schools, Cleveland;
Professor Vidal Arceo Tam, University of the Philippines;
Dr. Marian Marsh Torrey, Goucher College.

A committee consisting of Mr. S. A. Joffe and Professor Tomlinson Fort was appointed to audit the accounts of the Treasurer for the current year, and to report to the Board of Trustees at the Annual Meeting. A list of nominations of officers and other members of the Council was adopted and ordered printed on the official ballot for the annual election.

A committee was appointed to make arrangements for the Annual Meeting. It was announced that President Birkhoff had appointed the following members to represent the Society: Professor Richard Morris at the inauguration of President

J. M. Thomas of Rutgers University on October 14, 1925; Professor J. L. Gibson at the semi-centennial of Brigham Young University on October 16, 1925.

Professor Oswald Veblen reported that through the generosity of the General Education Board, the National Academy was able to distribute funds to aid in the printing of scientific periodicals and that for the fiscal year ending July 1926, eleven hundred dollars had been granted to the BULLETIN and two thousand dollars to the TRANSACTIONS OF THE SOCIETY.

Professor J. K. Whittemore presided at the morning session, and Professor H. W. Tyler in the afternoon.

Titles and abstracts of the papers read at this meeting follow below. The papers of Dickson, Ettlinger, Franklin (second paper), Gehman, Jackson, Kellogg, Raynor, Schwatt, Stone (first paper), Thomas, Walsh, and Wilder were read by title. Mr. Kormes was introduced by Professor Pfeiffer.

1. Professor H. P. Manning: *Definitions and postulates for relativity.*

This paper attempts to give definitions and postulates for relativity, in a space of one dimension. The undefined elements are positive and negative light particles, each class forming a linear continuous series of the type of the series of all real numbers. An event consists of two light particles, one from each series. A particle (not light particle) is a series of events determined by a one-to-one correspondence of the elements taken in order of the two given series. For a particle A the local time of any one of its events is the number assigned to that event in a correspondence of them all and the series of real numbers. This number is also assigned to each of the two light particles of the event and for them called t_1 and t_2 respectively. Thus the light particles are represented by these numbers, any event by a pair of them, and a particle by an equation. Finally, time and position according to A are defined, and the relation of two systems of this kind is worked out.

2. Mr. G. Y. Rainich: *Space-time and mass.*

In order to extend to more general cases the discussion of a previous paper establishing, for the centro-symmetric case, the connection between mass and curvature, four-dimensional spaces are studied which contain an infinity of three-dimensional spaces such that their Riemann tensors are particular cases of the Riemann tensors of the containing four-

dimensional space. The case is considered when the Riemann tensor determines in the three-dimensional spaces tensors of the second rank which are differentials of functions with zero divergence with a view of interpreting the residues of these functions around singularities corresponding to particles as masses of these particles.

3. Professor O. D. Kellogg: *Interpretations of Poisson's integral.*

A number of interpretations of Poisson's integral in two dimensions, such as those of Schwarz and Neumann, are well known. The present paper develops analogous results for Poisson's integral in space, and in addition gives several interpretations which appear to be new. By means of them various properties of harmonic functions of the sphere become more clearly evident and more easily proved.

4. Professor I. J. Schwatt: *The summation of a family of series of the type* $\Sigma_{k=1}^{q}(-1)^k([k/h])^p r^k$, *q either finite or infinite.*

In addition to the form given in the title of the paper, the value of summations like $\Sigma_{k=1}^{q}(-1)^k([k/h])^p \sin kx r^k$, $[k/h]$ being the largest integer in k/h, and similar forms are considered. The work involves several principles in the operation with series.

5. Professor A. A. Bennett: *Large primes have at least five consecutive quadratic residues.*

The primes are distributed among a complete system of nonoverlapping linear forms, and it is shown that for each prime greater than 149 there are at least five consecutive quadratic residues by actually exhibiting for each of them such a set. The discussion provides therefore something more than a mere existence proof. The method is elementary, but would probably involve an unreasonable amount of labor were it applied to still higher cases.

6. Mr. Mark Kormes: *A note on the functional equation* $f(x+y)=f(x)+f(y)$.

In this paper the following theorem is proved. Every solution of the functional equation $f(x+y)=f(x)+f(y)$ which is bounded on a set M of measure $m(M)>0$ has the form $A \cdot x$.

7. Dr. H. M. Gehman: *Some theorems on continuous curves containing no simple closed curve.*

The author proves the following theorems: (I) If $T+T'$ is a closed, totally disconnected subset of a bounded continuum M, and if T is the set

of all cut points of M, then M is a continuous curve containing no simple closed curve and T is the set of all end points of M. (II) If T is a closed, totally disconnected subset of a bounded continuum M, then M is a continuous curve containing no simple closed curve, with all its end points in T, if and only if no proper connected subset of M contains T. (III) A bounded continuum M is a continuous curve, if and only if every closed, totally disconnected subset T of M is a subset of some subcontinuum of M which contains no proper connected subset containing T. (IV) A closed subset M of a continuous curve S is a subset of a continuous curve containing no simple closed curve and lying in S, if and only if every maximal connected subset of M is a continuous curve containing no simple closed curve, and not more than a finite number of them are of diameter greater than any given positive number. (V) Every closed, bounded, totally disconnected point set is identical with the end points of some continuous curve containing no simple closed curve.

8. Dr. H. M. Gehman: *On irredundant sets of postulates.*

This paper will appear in full in an early issue of this BULLETIN.

9. Professor R. L. Wilder: *A property which characterizes continuous curves.*

Let M be a point set, C_1 and C_2 mutually exclusive closed subsets of M, and K a connected subset of M such that (1) K contains no points of the set C_1+C_2 and (2) both C_1 and C_2 contain limit points of K. Then K, together with its limit points in C_1+C_2, is called a set $K(C_1, C_2)M$. If M has the property that for every two of its mutually exclusive closed subsets C_1, C_2 and every connected set N containing points of both C_1 and C_2, there exists some set $K(C_1, C_2)M$ which has a point in common with N, then M is called normally connected. The following theorem is proved: *In order that a continuum M should be a continuous curve, it is necessary and sufficient that it be normally connected.*

10. Professor R. L. Wilder: *A theorem on connected point sets which are connected im kleinen.*

The following theorem is established: Let K and N be two mutually exclusive point sets, and M a connected, connected im kleinen point set which has at least one point in common with each of the sets K and N. Then there exists a point set H, a subset of M, such that H is connected and contains no point of either K or N, but such that K and N each contain at least one point of M which is a limit point of H.

11. Professor H. J. Ettlinger: *Note on the continuity of a function defined by a definite Lebesgue integral.*

The author shows that the following theorem recently proved by R. L. Jeffery (AMERICAN MATHEMATICAL MONTHLY, vol. 32 (1925), pp. 297-

299) can be obtained as a direct consequence of the Duhamel-Moore theorem: If (1) $f(x, y)$ is a real bounded function of (x, y) on the square $a \leqq x \leqq b$, $a \leqq y \leqq b$, and (2) $f(x, y)$ is a summable function of x on (a, b) for each y in (a, b), and (3) $f(x, y)$ is a continuous function of y at $\bar{y}$, for every x in (a, b), except a null set $M_{\bar{y}}$, then $F(y) = L\int_a^b f(x, y)\,dx$ is continuous at every point $\bar{y}$ in (a, b). The same result can also be obtained by the use of Lebesgue's theorem to the effect that the limit of a sequence of bounded summable functions is a summable function and the integral of the limit is the limit of the integral, where a null set of the interval may be disregarded. The above theorem may also be stated for m variables.

12. Professor J. L. Walsh: *On the expansion of an analytic function in a series of polynomials.*

This paper proves the following theorem, by means of results on conformal mapping due to Courant (GÖTTINGER NACHRICHTEN, 1914, pp. 101-109): If $f(z)$ is an analytic function of z in the interior of a Jordan curve C and if $f(z)$ is continuous in the closed region consisting of C and its interior, then throughout that closed region $f(z)$ can be expanded in a series of polynomials in z, the series converging uniformly in the same closed region.

13. Professor J. L. Walsh: *On the position of the roots of entire functions of genus zero and unity.*

This paper extends to entire function of genus zero and unity results formerly proved by the author for polynomials.

14. Professor G. E. Raynor: *On isolated singular points of harmonic functions.*

In the BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 52 (1924), Picard has given a characteristic of a harmonic function in the neighborhood of an isolated singular point in which neighborhood the function tends towards plus infinity. The purpose of the present paper is to characterize any harmonic function in the neighborhood of an isolated singular point and to obtain Picard's result as a special case and by a method different from his.

15. Professor W. A. Wilson: *On the structure of a limited continuum, irreducible between two points.*

Let ab denote the continuum, x one of its points, and X the oscillatory set about x. If neither of the saturated semi-continua X_a and X_b contained in $ab - X$ and containing a and b, respectively, is void, X is called *complete* if x is a common limiting point of X_a and X_b. (Likewise, A is complete of A_b is not void and a is a limiting point of A_b.) The necessary and suffi-

cient condition for ab to be the union of an aggregate of complete oscillatory sets without common points is that it contain no indecomposable continuum which is not a continuum of condensation. If $K=\{X\}$ denotes the aggregate of complete oscillatory sets of such a continuum, K has the order type of a linear segment $T=(0\leqq t\leqq 1)$. Furthermore the function $X=f(t)$ is an "upper semi-continuous function," i. e., $X=f(t)$ is the upper closed limit of $X'=f(t')$ as $t'\to t$ at each point of T.

16. Dr. T. H. Gronwall: *The algebraic structure of the formulas in plane trigonometry.* Third paper.

The present paper contains the complete solution of the derived equations of Hilbert, and the determination of the characteristic function, for the formulas of plane trigonometry.

17. Dr. T. H. Gronwall: *Summation of series and conformal mapping.*

Let $z=f(w)$ map $|w|<1$ on a simple region D in the z-plane, where D is interior to $|z|<1$, the boundary of D consists of a finite number of analytic curves, and $z=0$ for $w=0$, $z=1$ for $w=1$, and

$$w-1=c(z-1)^k+\cdots,$$

with $k>1$, at $z=1$. Moreover, let $g(w)$ be holomorphic and different from zero for $|w|<1$ and $g(w)\to 0$ as $w\to 1$ in an angle $<\pi$ with vertex at $w=1$ and interior to $|w|<1$, and let all coefficients in the expansion $1/g(w)=\Sigma c_n w^n$ be positive. The "sum" of the series

$$u_0+u_1+\cdots+u_n+\cdots$$

is now defined as $\lim_{n\to\infty} U_n$ (when existing), where U_n is defined by the formal identity

$$\Sigma_0^\infty u_n z^n=g(w)\ \Sigma_0^\infty c_n U_n w^n.$$

It is shown that when the series is summable (U) with the sum s, then $\Sigma u_n z^n\to s$ as $z\to 1$ in the angle referred to above. Under some additional restrictions on the order of magnitude of $g(w)$ when approaching a singularity on the unit circle, it is shown that when $u_0+u_1+\cdots+u_n$ is summable by Cesàro's means of *any* order, then it is summable (U) with the same sum; moreover, a very general comparison theorem for sums (U) with different values of $f(w)$ and $g(w)$ is established. The summation method of de la Vallée Poussin is the special case of (U) where

$$w=4z/(1+z)^2,\ g(w)=(1-w)^{1/2}.$$

18. Dr. T. H. Gronwall: *A new form of the remainder in the binomial series, with applications.*

Writing the binomial expansion of $(1-x)^{-k}$ with remainder term in the form

$$(1-x)^{-k}=\Sigma_0^{n-1}c_r x^r+x^n(1-x)^{-k}f(x),$$

where $k>0$, it is shown that for $0<x<1$, $f(x)$ is monotone increasing

for $k<1$, constant $(=1)$ for $k=1$, decreasing for $k>1$. When $0<k\leqq 1$, the coefficients in the power series for $f(x)$ are all positive, and

$$\left|(1-x)^{-k}-\Sigma_0^{n-1}c_\nu x^\nu\right|<|x|^n|1-x|^{-k}$$

for $|x|\leqq 1$ but $x\neq 1$. These formulas are very useful in the derivation of various asymptotic expansions.

19. Professor Philip Franklin: *Almost-periodic functions of two variables.*

In this paper we generalize the definition of almost-periodic functions of one variable, as given by Harold Bohr, to the case of two variables, and extend many of his results. We also obtain a few properties characteristic of the two variable case. As an example, we have the theorem that any almost-periodic function of two variables is, along any straight line, an almost-periodic function of the distance along that line.

20. Professor Philip Franklin: *The elementary character of certain integrals related to figures bounded by spheres and planes.*

In this note we show that the problem of calculating any surface, volume, or center of gravity of a surface or volume of a figure formed entirely from spheres and planes leads to an elementary integral, the result being an elementary function (combination of algebraic, trigonometric, and inverse trigonometric) of the parameters determining the figure.

21. Mr. W. L. Ayres: *Concerning the arcs and domains of a continuous curve.*

It is proved in this paper that if M and N are continuous curves and N is a subset of M, then, for any positive ϵ, $M-N$ contains not more than a finite number of maximal domains with respect to M of diameter greater than ϵ. It is shown that a continuous curve M cannot contain, for any positive ϵ, more than a finite number of arcs, having no common points except possibly end points and of diameter greater than ϵ, such that no point of any arc α, except its end points, is a limit point of $M-\alpha$. Also two theorems proved by R. L. Wilder are generalized.

22. Professor Norbert Wiener: *A new method in periodogram analysis.*

The author develops the theory of a new method of periodogram analysis, connected with the correlation-coefficient method of G. I. Taylor for the analysis and description of irregular sequences of data.

23. Dr. M. H. Stone: *The convergence of Bessel's series.*

An extension of Birkhoff's method of discussing expansion problems is here applied to Bessel's series. The leading results of W. H. Young (PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY, 1920) are thus demonstrated by an independent treatment for developments in terms of the functions $J_\nu(\lambda_k x)$ where $\nu > -\frac{1}{2}$, $J_\nu(\lambda_k) = 0$.

24. Dr. M. H. Stone: *The Borel summability of Fourier series.*

The Borel summability of Fourier series is discussed by means of an integral of the form $\int_0^A e^{-\alpha} f(x, \alpha) d\alpha$, in which $A \to \infty$. C. N. Moore has announced results showing the comparative ineffectiveness of this sum. On introducing a convergence factor, the integral becomes $\int_0^A (1-\alpha/A) e^{-\alpha} f(x,\alpha) d\alpha$. It is shown that as $A \to \infty$ the integral converges for almost all values of x to the function for which the series is defined. It is found possible to obtain similar results by applying a modification of the Borel sum to the Green's function for the differential system $u' + \lambda u = 0$, $u(0) - u(1) = 0$.

25. Dr. T. Y. Thomas: *The identities of affinely connected manifolds.*

In this paper the idea of a complete set of identities of an invariant is introduced as the set of algebraically independent identities furnishing all the algebraic conditions on the invariant. Hence every identity satisfied by the invariant can be deduced by algebraic processes from the identities of the complete set. The paper gives the complete set of identities of the affine and metric normal tensors which constitute complete sets of invariants of general affinely connected manifolds and Riemann spaces respectively. In addition the completeness proof for the identities satisfied by the curvature tensor $B^i_{\alpha\beta\gamma}$ in the general manifold, and for the covariant form of the curvature tensor $B_{\alpha\beta\gamma\delta}$ in the Riemann geometry is given.

26. Professor M. J. Babb: *The research manuscripts and library of Dr. Robert Adrain, professor of mathematics at Rutgers, Columbia, and Pennsylvania.* Preliminary report.

In this paper, a brief description is given of the manuscripts, books, and portrait of Robert Adrain recently presented to the University of Pennsylvania by his great-grandchildren through Frederick S. Duncan, Esq., of Englewood, N. J., together with the records and correspondence that have recently come to light at that University.

27. Professor Dunham Jackson: *Note on the convergence of Fourier series.*

In a recent number of the AMERICAN MATHEMATICAL MONTHLY (December, 1924), Franklin has given a very clear and simple proof of the convergence of Fourier series under hypotheses sufficiently general to cover the most important applications. The purpose of this note, starting with the same hypotheses, is to present a proof which is believed to be materially simpler still.

28. Professor L. E. Dickson: *New division algebras.*

Only one type of division algebras is known. This paper obtains an infinitude of new types, one for each solvable group. The paper has been offered for publication in the TRANSACTIONS.

29. Dr. H. M. Gehman: *A theorem on continuous curves in space of n dimensions.*

The author has proved that any continuous curve containing no simple closed curve and lying in a space of n dimensions is in continuous (1, 1) correspondence with some continuous curve of the same type in space of two dimensions. This answers a question proposed by S. Mazurkiewicz (FUNDAMENTA MATHEMATICAE, vol. 2 (1921), p. 130).

R. G. D. RICHARDSON,
Secretary.

THE CHAUVENET PRIZE FOR MATHEMATICAL EXPOSITION

In March, 1925, President J. L. Coolidge proposed that the Mathematical Association of America establish a prize for special merit in mathematical exposition. The proposition was sanctioned by mail vote of the Trustees and a committee consisting of Professors A. J. Kempner, Chairman, Louise D. Cummings and D. R. Curtiss was appointed to formulate the details. This committee presented a report at the Ithaca meeting which in somewhat modified form, was adopted by the Trustees. The substance of the report is as follows.

The committee believe that the proposed prize will exert a desirable influence on the production of high-grade expository articles. They adopted the name suggested by President Coolidge, namely, "The Chauvenet Prize for Mathematical Exposition." For a study of the life and influence of William Chauvenet, 1820-1870, Professor of mathematics in the U. S. Navy, 1847-1859, President of the Academic Board of the Navy, 1847-1850, Professor of mathematics and natural philosophy at Washington University, St. Louis, Mo., 1859-1869, author of many works and treatises, they refer to an article by Professor W. H. Roever, in WASHINGTON UNIVERSITY STUDIES, Vol. 12, SCIENTIFIC SERIES, No. 2, 1925.

The Chauvenet Prize is to be awarded every five years for the best article of an expository character dealing with some mathematical topic, written by a member of the Mathematical Association of America and published in English in a journal during the five calendar years preceding the award. This prize will not be awarded for books, even though a large portion of mathematical books are mainly or completely expository in character, such as textbooks. They bring their own reward in the form of royalties.

The amount of the prize was fixed at one hundred dollars, an amount which the committee deemed sufficiently large to

be attractive apart from the honor of the award. The cash for the first award has been provided by a friend of the Association. Thereafter, it will be supplied from the Association treasury, one-fifth of the amount being set aside each year for the purpose.

The first award is to be made at the annual meeting in December, 1925, covering the five-year period ending with the calendar year 1924.*

It is provided that the award shall be determined at each quinquennial period by a scrutinizing committee of three to be appointed by the president of the Association and that this committee should be restricted as little as possible, aside from the specifications mentioned in the foregoing paragraphs. President Coolidge appointed the scrutinizing committee for the first award as follows: Professors W. C. Graustein, Anna J. Pell-Wheeler, E. B. Van Vleck (chairman).

THE INTERNATIONAL CONFERENCE FOR THE USE OF ESPERANTO IN SCIENCE†

The international conference on the use of Esperanto in science, held in Paris on May 14, 15, 16, 1925, simultaneously with the conference on its use in commerce, signalizes the restoration of this international movement, which was stopped almost completely during the war. Among the members of the French Academy of Sciences who lent their support to this conference are the physiologist Richet, the physicists

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* The first award of this prize was made since this article was written, at the Kansas City Meeting of the Association. The recipient was Professor G. A. Bliss, of the University of Chicago. The paper which was the basis of the award was his paper *Algebraic functions and their divisors*, published in the ANNALS OF MATHEMATICS for September and December, 1924.

† A report prepared by Professor M. Fréchet, who attended the Congress as the official delegate of this Society.

Cotton, Guillaume, Janet, the astronomer Bigourdan and the mathematicians Appell and Painlevé, then Prime Minister of France.

There were present representatives of a number of governments, of academies (among them those of Washington and of Maryland) and of engineering societies; more than 150 scientists from 20 different countries were registered. Professor Cotton was elected President of the conference. The meetings were conducted in Esperanto. The following program was adopted:

To recommend the teaching of Esperanto in public schools and the granting of special advantages in examinations to candidates who can use Esperanto.

To adopt Esperanto as one of the authorized languages of scientific congresses.

To recommend the publication of international technical vocabularies.

To affiliate with the "Internacia Scienca Associo."*

I wish to thank the American Mathematical Society for having given me the honor of appointing me as their delegate at the conference.

MAURICE FRÉCHET

UNIVERSITY OF STRASBOURG.

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* This International Association for the use of Esperanto in Science admits individual members at annual dues of $2.50 (life membership at $40) and corporate membership at $5.00 (permanent membership at $75). The Secretary is M. Célestin Rousseau, 4 Place de la République, Levallois-Péret, Seine, France. The Esperanto-Associo de Nord Ameriko has offices at 507 Pierce Building, Copley Square, Boston 17, Mass.

THE HEAVISIDE OPERATIONAL CALCULUS*

BY J. R. CARSON

The Heaviside operational calculus is a systematic method, originated and developed by Oliver Heaviside, for the solution of systems of linear differential equations with constant coefficients, and linear partial differential equations of the type of the *wave equation*. Its important applications in mathematical physics are to the dynamics of small oscillations, the Fourier theory of heat flow and to electrical transmission theory. It was, in fact, in connection with problems of the latter class that the operational calculus was developed, and in the solution of such problems it is an instrument of remarkable directness, simplicity and power.†

The operational calculus may be described with sufficient generality for our present purposes, in connection with the solution of the system of linear differential equations

$$(1)\qquad \begin{cases} a_{11}x_1+a_{12}x_2+a_{13}x_3+\cdots+a_{1n}x_n=f_1(t)\ , \\ a_{21}x_1+a_{22}x_2+a_{23}x_3+\cdots+a_{2n}x_n=f_2(t)\ , \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+a_{n3}x_3+\cdots+a_{nn}x_n=f_n(t)\ , \end{cases}$$

where the coefficient a_{jk} has the form

$$(2)\qquad a_{jk}=\alpha_{jk}+\beta_{jk}\frac{d}{dt}+\gamma_{jk}\frac{d^2}{dt^2}+\cdots .$$

$\alpha, \beta, \gamma, \cdots$ being constants. We require the solution for $x_1, x_2, \cdots, x_n$ for the following boundary conditions: The known functions $f_1, \cdots, f_n$ and the dependent variables $x_1, \cdots, x_n$ are identically zero for $t<0$. In the language of dynamics, the "forces" $f_1, \cdots, f_n$ are applied to a system

* An address delivered before the Society, May 2, 1925.

† For a brief discussion of the Heaviside operational calculus from the point of view adopted in this paper, see Carson, *The Heaviside Operational Calculus*, BELL SYSTEM TECHNICAL JOURNAL, Nov., 1922.

characterized by the coordinates $x_1, \cdots, x_n$, when in a state of equilibrium. These boundary conditions are extremely important to the physicist, and it is to the solution of this boundary value problem that the operational calculus applies.

In view of the linear character of the equations, we may, without loss of essential generality, set all the f functions equal to zero except one, say $f_1(t)$, and write the equations as

$$(3) \qquad \begin{cases} a_{11}x_1+a_{12}x_2+ \cdots +a_{1n}x_n=f_1(t) , \\ a_{21}x_1+a_{22}x_2+ \cdots +a_{2n}x_n=0 , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}x_1+a_{n2}x_2+ \cdots +a_{nn}x_n=0 . \end{cases}$$

Now consider the auxiliary system of equations

$$(4) \qquad \begin{cases} a_{11}h_1+a_{12}h_2+ \cdots +a_{1n}h_n=1 , \quad t\geqq 0 , \\ a_{21}h_1+a_{22}h_2+ \cdots +a_{2n}h_n=0 , \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ a_{n1}h_1+a_{n2}h_2+ \cdots +a_{nn}h_n=0 , \end{cases}$$

in the auxiliary variables $h_1, \cdots, h_n$. As in (1) and (2), both sides of the equations are supposed to be identically zero for $t<0$.

For the equilibrium boundary conditions the solution of (4) supplies the solution of (3) by virtue of the formula

$$(5) \qquad x_i=\frac{d}{dt}\int_0^t f(t-\tau)h_i(\tau)d\tau , \qquad (i=1, 2, \cdots, n) .$$

This formula is easily established in a number of ways; perhaps the simplest, and at the same time the most general, method of proof is to regard $f(t)$ as the limit of a summation of the form

$$\sum \phi(t-n\delta t)\delta f_n ,$$

where $\phi(t)$ is defined as a function which is zero for $t<0$ and unity for $t\geqq 0$. The solution of (4) applies to this case and passing to the limit ($\delta t \to 0$) formula (5) results by usual processes.

It is to equations (4) that the operational calculus applies; Heaviside did not explicitly consider equations (3) in their completely general form.

Heaviside's treatment of equations (4) proceeded as follows: Replacing the differential operator d^n/dt^n by p^n, the differential equations become formally algebraic and yield the *operational equation*

$$h_i=\frac{1}{H_i(p)}\,, \qquad (i=1, 2, 3, \cdots, n)\,. \tag{6}$$

I have called (6) an equation; as a matter of fact it is not an equation except in a purely symbolic sense, since the left hand side is a function of t and the right hand side a function of p. Heaviside's point of view, however, was that (6) is the full equivalent of (4), and that the functional form of $H(p)$ contains all the information necessary for the explicit solution.

From this point on, however, his method was purely inductive; from the known solution of specific problems, he inferred rules for expanding and interpreting the operational equation (6), and for converting it into the explicit solution. The body of rules and formulas so arrived at is termed the operational calculus.

Instead of following Heaviside's inductive methods, we shall now take advantage of the following theorem.

The operational equation

$$h_i=1/H_i(p)\,.$$

is simply the symbolic or short hand equivalent of the integral equation

$$\frac{1}{pH_i(p)}=\int_0^\infty h_i(t)e^{-pt}dt\,,$$

which is valid for all values of p in the right hand half of the complex plane. This uniquely determines a continuous function $h_i(t)$ in terms of $H_i(p)$. From this integral equation the rules and formulas of the operational calculus are directly and immediately deducible.

The equation (dropping subscripts)

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt \tag{7}$$

is an integral equation of the Laplace type. Beyond the proof that it determines a unique continuous function $h(t)$, curiously little work appears to have been done on it from the standpoint of analysis. At present, however, we are concerned with it only as an instrument for the solution of the differential equations (4).

The proof of the preceding theorem is easily arrived at as follows: Referring to equation (3) suppose that $f(t) = e^{pt}(t \geqq 0)$ where p *is a positive real constant* or complex with real part positive. The solution of equations (3) with the subscripts dropped) is then of the form

$$x = \frac{e^{pt}}{H(p)} + y(t) , \tag{8}$$

where $y(t)$ is the complementary solution, so constructed as to satisfy the equilibrium boundary condition. But by (5), we have also

$$x = \frac{d}{dt}\int_0^t e^{p(t-\tau)}h(\tau)d\tau , \tag{9}$$

whence, by direct equation,

$$\frac{e^{pt}}{H(p)} + y(t) = \frac{d}{dt}\int_0^t e^{p(t-\tau)}h(\tau)d\tau .$$

Carrying out the indicated differentiation and simplifying, we get

$$\frac{1}{H(p)} + y(t)e^{-pt} = h(t)e^{-pt} + p\int_0^t h(\tau)e^{-\tau p}d\tau .$$

Finally, setting $t = \infty$,

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt ,$$

provided the real part of p is positive. This establishes formula (7) as an identity for all values of p in the right hand half of the complex plane.

Before proceeding with an application of the foregoing integral equation, it should be remarked that while the operational calculus has been elucidated in connection with a set of differential equations involving a finite number of variables, it is not so limited in its applications. The derivations of formula (5) and the integral equation (7) are quite general and depend only on the linear character of the differential equations and the linear invariable character of the dynamic system which they describe. Consequently the resulting formulas are applicable also when the number of variables is infinite (cases of which occur in important electrotechnical problems) and to linear partial differential equations of the type of the *wave equation*, the *equation of the conduction of heat*, and the *telegraph equation*.*

We now turn to a discussion of the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t)e^{-pt}dt$$

as a means of deriving the rules of the operational calculus, and as an instrument for the direct solution of problems. My own work on this problem has been directed as follows: (1) guided by the inductive work of Heaviside on the operational equation, to develop general rules for the expansion and transformation of the integral equation, leading to general types of expansion solutions. (2) The accumulation of a table of definite integrals of the type

$$\int_0^\infty \phi(t)e^{-pt}dt\,.$$

It hardly needs to be pointed out, in view of the symbolic equivalence of the operational and the integral equations, that

* See Whittaker and Watson, *Modern Analysis*, p. 386, for a discussion of these equations.

the evaluation of an infinite integral of this type immediately furnishes the solution of the corresponding operational equation. The table of integrals which has been collected has proved quite valuable in technical problems.

We shall now deduce two expansion solutions of the integral equation, corresponding to two important forms of solution of the operational equation first given by Heaviside.

The Power Series Solution. Let us suppose, as is the case in a very large class of physical problems, that the function $1/H(p)$ admits of formal asymptotic expansion; thus

$$1/H(p) \sim \sum_0^\infty a_n/p^n . \tag{10}$$

With the convergence or divergence of such an expansion we have no concern. It must, however, satisfy Poincaré's definition of an asymptotic expansion.*

Now starting with the integral equation (7) and integrating by parts, we get

$$\frac{1}{H(p)} = h(0) + \int_0^\infty e^{-pt} h^{(1)}(t)dt ,$$

where $h^{(n)}(t) = (d^n/dt^n)h(t)$. Now let p approach infinity; in the limit the integral vanishes and, by virtue of the asymptotic expansion (10), the left hand side becomes a_0. Consequently

$$h(0) = 1/H(\infty) = a_0 .$$

Integrating again by parts, we get

$$p\left(\frac{1}{H(p)} - a_0\right) = h^{(1)}(0) + \int_0^\infty e^{-pt} h^{(2)}(t)dt .$$

Now let p again approach infinity; in the limit the integral vanishes and the left hand side by virtue of (10) becomes a_1. Consequently

$$h^{(1)}(0) = a_1 .$$

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* See Whittaker and Watson, *Modern Analysis*, p. 151.

Proceeding in this way we establish the relation

$$h^{(n)}(0)=a_n.$$

It follows at once that, if we *assume* that a power series expansion of $h(t)$ exists, it is given by

$$h(t)=\sum_0^\infty a_n t^n/n! \tag{11}$$

We thus arrive at the Heaviside rule:

In the operational equation,

$$h=1/H(p),$$

expand $1/H(p)$ *in inverse powers of* p:

$$1/H(p)\sim a_0+a_1/p+a_2/p^2+\cdots. \tag{10}$$

The explicit solution for $h(t)$ *results by replacing* $1/p^n$ *by* $t^n/n!$ *in the asymptotic expansion, so that*

$$h(t)=a_0+a_1t/1!+a_2t^2/2!+\cdots \tag{11}$$

*is the required power series solution.**

It may be remarked in passing that this type of solution, while extremely direct and always possible in the case of systems having a finite number of degrees of freedom, is of minor practical importance unless the power series can be recognized and summed. Furthermore this form of solution does not exist in many important technical problems.

The Solution in Terms of Normal Vibrations. It is known from the usual elementary theory of linear differential equations that the solution of equations (4) may be written in the form

$$h_i=C_{i0}+\sum_0^m C_{ij}e^{p_jt}, \qquad (i=1, 2, \cdots, n) \tag{12}$$

where $p_1, p_2, \cdots, p_m$ are the roots of the equation

$$H_i(p)=\frac{D(p)}{M_i(p)}=0, \tag{13}$$

* It is interesting to note that the series (11) is Borel's associated function of the series (10), and that the infinite integral is the Borel sum of the series (10).

and the C's are constants of integration which must be so chosen as to satisfy the system of differential equations and the imposed boundary conditions. Assuming that these roots can be located, the problem of determining the constants of integration, while formally straightforward, is extremely troublesome in practice when the number of degrees of freedom is large. It will now be shown that, assuming the roots $p_1, p_2, \cdots$ to have been evaluated, the constants of integration can be evaluated from the integral equation with extreme simplicity. Dropping the double subscript, we write the known form of solution as

$$h = C_0 + \sum C_j e^{p_j t}, \tag{14}$$

and substitute in the integral equation

$$\frac{1}{pH(p)} = \int_0^\infty h(t) e^{-pt} dt. \tag{15}$$

Integrating term by term we get

$$\frac{1}{pH(p)} = \frac{1}{p} C_0 + \sum \frac{C_j}{p - p_j}, \tag{16}$$

where, it will be recalled, p_j is a root* of $H(p)$.

Multiplying through by $pH(p)$ we get

$$C_0 H(p) + \sum \frac{pH(p)}{p - p_j} C_j = 1. \tag{17}$$

We now introduce restrictions which obtain in physical problems: $H(p)$ has no zero root, no repeated roots, and $1/H(p)$ is a proper fraction of the form $M(p)/D(p)$ where the numerator and denominator are prime to each other, and the numerator of lower order in p than the denominator.

Now if we set $p = 0$, the summation vanishes, and we get

$$C_0 = 1/H(0).$$

* In all physical problems, the real part of p_j is negative, so that the integration of (15) is valid.

Next set p equal to p_j+q, and then let q approach zero. Since $H(p_j)=0$, it follows at once that (17) reduces to

$$p_jH'(p_j)C_j=1\,,$$

whence

$$C_j=\frac{1}{p_jH'(p_j)}\,, \tag{18}$$

where

$$H'(p_j)=\left[\frac{d}{dp}H(p)\right]_{p=p_j}.$$

We thus arrive at the solution

$$h(t)=\frac{1}{H(0)}+\sum\frac{e^{p_jt}}{p_jH'(p_j)}\,, \tag{19}$$

first stated by Heaviside without proof. It was apparently arrived at inductively from the operational equation

$$h=1/H(p)\,.$$

This is a beautifully compact solution, and of first-rate importance in many technical problems.

It is quite beyond the scope of this paper to discuss in detail the appropriate expansion and transformations of the integral equations, each having its counterpart in the corresponding operational equation, which are useful when the operational equation does not admit of direct solution as it stands. A few theorems, immediately derivable from the integral equation, may be of interest.

1. *If two functions h and g are defined by the operational equations*

$$h=1/H(p)\,,\quad \textit{and}\quad g=1/pH(p)\,,$$

then

$$g(t)=\int_0^t h(t)dt\,.$$

2. *If h and g are defined by the operational equations*

$$h=1/H(p)\,,\qquad g=p/H(p)\,,$$

then

$$g(t)=\frac{d}{dt}h(t)\ ,$$

provided $h(0)=0$.

3. *If h and g are defined by the operational equations*

$$h=1/H(p)\ , \qquad g=1/H(p+\lambda)\ ,$$

then

$$g(t)=\left(1+\lambda\int_0^t dt\right)e^{-\lambda t}h(t)\ .$$

4. *If h and g are defined by the operational equations*

$$h=1/H(p)\ , \qquad g=e^{-\lambda p}/H(p)\ ,$$

then

$$g=0 \quad for \quad t<\lambda\ ,$$
$$=h(t-\lambda) \quad for \quad t\geqq\lambda\ .$$

5. *Let h be defined by the operational equation*

$$h=1/H(p)\ ,$$

and suppose that $H(p)$ *can be factored in the form*

$$H(p)=H_1(p)\cdot H_2(p)\ .$$

Let the auxiliary functions h_1 *and* h_2 *be defined and determined by the auxiliary operational equations*

$$h_1=1/H_1(p)\ , \qquad h_2=1/H_2(p)\ ;$$

then

$$h(t)=\frac{d}{dt}\int_0^t h_1(t-\tau)h_2(\tau)d\tau$$
$$=\frac{d}{dt}\int_0^t h_2(t-\tau)h_1(\tau)d\tau\ .$$

The following theorem is of considerable theoretical interest, as it formally extends the methods of the operational

calculus to the case of arbitrary functions $f(t)$ as compared with the "unit" function of equations (4).

6. *If the function $f(t)$ of equations* (3) *is such that the infinite integral*

$$\int_0^\infty f(t)e^{-pt}dt$$

can be evaluated and is equal to $1/pF(p)$, then $x(t)$ of equations (3) *is given symbolically by the operational equation*

$$x = \frac{1}{F(p) \cdot H(p)} = \frac{1}{X(p)},$$

and by the corresponding integral equation

$$\frac{1}{X(p)} = \int_0^\infty x(t)e^{-pt}dt .$$

This integral equation corresponds precisely with (7), *and is solvable by the same general procedure.*

The Telegraph Equation. The *telegraph equation* formulates the propagation of current and voltage along the transmission line. It will serve as an excellent example of the utility of the operational calculus and the integral equation.

We suppose that the transmission line extends along the positive axis and is energized by an electromotive force applied at $x=0$. The differential equations of current I and voltage V are

$$\left(L\frac{d}{dt}+R\right)I = -\frac{\partial}{\partial x}V ,$$

$$\left(C\frac{d}{dt}+G\right)V = -\frac{\partial}{\partial x}I ,$$

where L, R, C and G are the distributed constants of the line, the inductance, resistance, capacity and leakage per unit length.

Replacing d/dt by the symbol p, we get

$$(Lp+R)I=-\frac{\partial}{\partial x}V\ ,$$

$$(Cp+G)V=-\frac{\partial}{\partial x}I\ ,$$

thus formally eliminating the variable time t. Successive elimination of V and of I give the equations

$$\gamma^2 I=\frac{\partial^2}{\partial x^2}I\ ,$$

$$\gamma^2 V=\frac{\partial^2}{\partial x^2}V\ ,$$

where $\gamma^2=(Lp+R)(Cp+G)$. These equations are satisfied by solutions of the form

$$V=Ae^{-\gamma x}+Be^{\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}(Ae^{-\gamma x}-Be^{\gamma x})\ ,$$

where A and B are constants of integration.

We now assume that the line is infinitely long so that the reflected wave vanishes. We also assume a voltage $V_0(t)=V_0$ applied at $x=0$. The equations then become

$$V=V_0e^{-\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}V_0e^{-\gamma x}\ .$$

Finally, if we assume that V_0 is zero for $t<0$ and unity for $t\geqq 0$, we have the *operational equations*

$$V=e^{-\gamma x}\ ,$$

$$I=\frac{Cp+G}{\gamma}e^{-\gamma x}\ .$$

Let us write

$$\gamma^2=\frac{1}{v^2}[(p+\rho^2)-\sigma^2]\,,$$

where

$$v^2=1/LC\,,\quad \rho=\frac{R}{2L}+\frac{G}{2C}\,,\quad \sigma=\frac{R}{2L}-\frac{G}{2C}\,,$$

and let us consider the operational equation defining a new variable F:

$$F=\frac{p}{v\gamma}e^{-\gamma x}=\frac{p}{\sqrt{(p+\rho)^2-\sigma^2}}e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}\,. \tag{20}$$

It follows at once from the operational rules discussed above (Theorems 1 and 2) that

$$I=v\left(C+G\int_0^t dt\right)F\,, \tag{21}$$

$$V=-v\int_0^t\frac{\partial F}{\partial x}\,dt\,. \tag{22}$$

Our problem is thus reduced to evaluating the function F, as defined by the operational equation (20) and the corresponding integral equation

$$\frac{e^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}=\int_0^\infty F(t)e^{-pt}dt\,. \tag{23}$$

The solution of this integral equation is deducible from the known definite integral

$$\int_\lambda^\infty e^{-pt}J_0(\sqrt{t^2-\lambda^2})dt=\frac{e^{-\lambda\sqrt{p^2+1}}}{\sqrt{p^2+1}}\,. \tag{24}$$

This, regarded as an integral equation, determines a function which is zero for $t<\lambda$ and has the value $J_0(\sqrt{t^2-\lambda^2})$ for $t\geqq\lambda$, J_0 being the Bessel function of the first kind of order zero. In order to solve (23) by aid of (24) we proceed to transform the latter as follows:

(1) Let $\lambda p=q$ and $t/\lambda=t_1$. Substitute in (24) and then replace q by p and t_1 by t, in order to retain our original symbolism. We get

$$\frac{\epsilon^{-\sqrt{p^2+\lambda^2}}}{\sqrt{p^2+\lambda^2}}=\int_1^\infty \epsilon^{-pt}J_0(\lambda\sqrt{t^2-1})dt\ . \tag{25}$$

(2) Making the substitution $p=q+\mu$ in (25) and then replacing q by p, we get

$$\frac{\epsilon^{-\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}=\int_1^\infty \epsilon^{-pt}\epsilon^{-\mu t}J_0(\lambda\sqrt{t^2-1})dt\ . \tag{26}$$

(3) Making the substitution

$$p=\frac{x}{v}q\ ,\ \text{and}\ t=\frac{v}{x}t_1\ ,$$

in (26), and ultimately replacing q by p and t_1 by t, we get

$$\frac{\epsilon^{-\frac{x}{v}\sqrt{(p+\mu)^2+\lambda^2}}}{\sqrt{(p+\mu)^2+\lambda^2}}=\int_{x/v}^\infty \epsilon^{-pt}\epsilon^{-\mu t}J_0(\lambda\sqrt{t^2-x^2/v^2})dt \tag{27}$$

where λ and μ are parameters at our disposal, except that, as yet, they are restricted to positive real values.

(4) Now if we compare (27) with the integral equation (23) it will be observed that the left sides become identical if we set $\mu=\rho$ and $\lambda=i\sigma=\sigma\sqrt{-1}$. (This last operation is justified because $\sigma\leqq\rho$.) Introducing these relations we get finally

$$\frac{\epsilon^{-\frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}}}{\sqrt{(p+\rho)^2-\sigma^2}}=\int_{x/v}^\infty \epsilon^{-pt}\epsilon^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2})dt\ , \tag{28}$$

where $I_0(x)=J_0(ix)$. Consequently it follows that

$$(29)\quad \begin{cases} F=0 \quad \text{for} \quad t<x/v \\ =e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}) \quad \text{for} \quad t\geqq x/v\,. \end{cases}$$

It now follows from (21), (22) and (29) by straightforward procedure, that

$$(30)\quad \begin{cases} I=0 \quad \text{for} \quad t<x/v \\ =\sqrt{\dfrac{C}{L}}e^{-\rho t}I_0(\sigma\sqrt{t^2-x^2/v^2}) \\ \quad +vG\displaystyle\int_{x/v}^{t} e^{-\rho\tau}I_0(\sigma\sqrt{\tau^2-x^2/v^2})\,d\tau\,, \qquad t\geqq x/v; \end{cases}$$

$$(31)\quad \begin{cases} V=0 \quad \text{for} \quad t<x/v \\ =e^{-\rho x/v}+\dfrac{\sigma x}{v}\displaystyle\int_{x/v}^{t}\frac{e^{-\rho t}I_1(\sigma\sqrt{\tau^2-x^2/v^2})}{\sqrt{\tau^2-x^2/v^2}}\,d\tau, \qquad t\geqq x/v. \end{cases}$$

The foregoing is believed to be an excellent example of the value of the operational calculus and in particular of the advantage attaching to the recognition of the integral equation identity. The directness and simplicity of the solution, as derived above, as compared with its derivation by classical methods, is noteworthy. At the same time it must be admitted that a direct solution from the operational equation, without recognizing the integral equation identity, presents formidable difficulties. Heaviside's own attack on this problem from the operational equation, while distinguished by extraordinary ingenuity and almost uncanny intuition, can hardly be regarded as entirely satisfactory.*

However, without a knowledge of the integral identity (24) on which the preceding solution is based, it is possible to derive a series solution as follows. The method will be sketched for the operational equation

$$V=e^{-\gamma x}$$

* See Heaviside's *Electromagnetic Theory*, vol. II, p. 290 et seq.

for the voltage. The procedure for the current is the same and will require no explanation. The method of solution depends on Theorem 4, given above. Since

$$\gamma x = \frac{x}{v}\sqrt{(p+\rho)^2-\sigma^2}$$

$$= \frac{x}{v}\,p\sqrt{\left(1+\frac{\rho}{p}\right)^2-\frac{\sigma^2}{p^2}},$$

we can write the operational equation as

$$V = \epsilon^{-\frac{x}{v}p\sqrt{\left(1+\frac{\rho}{p}\right)^2-\frac{\sigma^2}{p^2}}}$$

$$= \epsilon^{-\frac{x}{v}p\left(1+\frac{\rho}{p}+\frac{a_1}{p^2}+\frac{a_2}{p^3}+\cdots\right)},$$

by the binomial expansion of the exponent. This is formally reducible to the form

$$V = \epsilon^{-\frac{x}{v}p}\cdot\epsilon^{-\frac{\rho x}{v}}\left(1+\frac{A_1}{p}+\frac{A_2}{p^2}+\frac{A_3}{p^3}+\cdots\right).$$

A direct application of Theorem 4 now gives

$$V = 0 \text{ for } t < x/v$$

$$= \epsilon^{-\frac{\rho}{v}x}\left\{1+A_1\frac{t-x/v}{1!}+A_2\frac{(t-x/v)^2}{2!}+A_3\frac{(t-x/v)^3}{3!}+\cdots\right\}.$$

It is easily verified that this series solution in the "retarded" time $t-x/v$ is absolutely convergent and identical with the expansion of the solution (21).

The Asymptotic Solution of Operational Equations. An extremely interesting and important part of the operational calculus relates to the derivation of asymptotic expansions directly from the operational equation. A study of the many problems for which Heaviside obtains asymptotic expansions

shows that they may be divided into two classes: (1) those of which the operational equation is of the form

$$\text{(I)} \qquad I=\sqrt{p}F(p)\ ,$$

and (2) those of which the operational equation is of the form*

$$\text{(II)} \qquad h=\phi(\sqrt{p})\ .$$

Heaviside himself gives no justification or proof of his expansion solutions. He does not formally distinguish between the two classes, and he gives no information regarding the asymptotic character of the series. While a completely satisfactory theory of these expansions has not as yet been worked out, the application of the Laplace integral equation to their investigation throws a great deal of light on the problem, and at least reduces it to a form to which the orthodox processes of analysis are applicable.

We start with the type of problem (class 1), which is symbolically formulated by operational equations of the type

$$\text{(32)} \qquad h=\sqrt{p}\ F(p)\ ,$$

of which the corresponding integral equation is

$$\text{(33)} \qquad \frac{F(p)}{\sqrt{p}}=\int_0^\infty h(t)e^{-pt}dt\ ,$$

and we suppose that $F(p)$ admits of the power series expansion

$$\text{(34)} \qquad a_0+a_1p+a_2p^2+\cdots\ .$$

For this problem the Heaviside rule is as follows.

If the operational equation

$$h=1/H(p)$$

* More generally $h=\phi(p^k\sqrt{p})$ where k is a positive integer. The case where $k=0$, however, illustrates Heaviside's procedure with sufficient generality.

can be expanded in the form

$$h=(a_0+a_1p+a_2p^2+a_3p^3+\cdots)\sqrt{p}\ ,$$

the explicit solution, in the form of an asymptotic series, is obtained by replacing the symbol $\sqrt{p}$ *by* $1/\sqrt{\pi t}$, *and* p^n *by* d^n/dt^n, *whence*

$$h(t)\sim\left(a_0+a_1\frac{d}{dt}+a_2\frac{d^2}{dt^2}+\cdots\right)\frac{1}{\sqrt{\pi t}} \tag{35}$$

$$\sim\frac{1}{\sqrt{\pi t}}\left(a_0-a_1/2t+1\cdot 3a_2/(2t)^2-1\cdot 3\cdot 5a_3/(2t)^3+\cdots\right). \tag{35a}$$

We now proceed to derive this rule from the integral equation (33), and to discuss its limitations.

We assume the existence of an auxiliary function $f(t)$, defined and determined by the auxiliary integral equation

$$F(p)=\int_0^\infty f(t)e^{-pt}dt\ . \tag{36}$$

Now since

$$\frac{1}{\sqrt{p}}=\int_0^\infty e^{-pt}\frac{dt}{\sqrt{\pi t}}\ ,$$

it follows from Theorem 5 above that $h(t)$, as defined by (33), is given by

$$h(t)=\frac{1}{\sqrt{\pi}}\int_0^t\frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\ . \tag{37}$$

We observe that if (36) is differentiated repeatedly with respect to p and p set equal to zero, then, by virtue of the expansion (34),

$$a_n=(-1)^n\int_0^\infty\frac{t^n}{n!}f(t)dt\ ,\qquad (n=1,\ 2,\ \cdots)\ . \tag{38}$$

It will be noted, in this connection, that this expansion presupposes the convergence of the infinite integrals (38) for all values of n and therefore imposes severe restrictions on $f(t)$ and hence on $F(p)$. We shall suppose that these restrictions are satisfied.

We are now prepared to prove the following theorem.

A necessary and sufficient condition for the validity of the Heaviside asymptotic expansion (35) *of the operational equation*

$$h=F(p)\sqrt{p}\ ,$$

is that the definite integral (37) *shall be asymptotically representable by the series*

$$(39)\qquad \frac{1}{\sqrt{\pi t}}\left\{\int_0^\infty f(t)dt+\frac{1}{2t}\int_0^\infty \frac{t}{1!}f(t)dt+\frac{1\cdot 3}{(2t)^2}\int_0^\infty \frac{t^2}{2!}f(t)dt+\frac{1\cdot 3\cdot 5}{(2t)^3}\int_0^\infty \frac{t^3}{3!}f(t)dt+\cdots\right\}.$$

This theorem is an immediate consequence of the preceding analysis, for by (38) the series (39) is simply

$$\frac{1}{\sqrt{\pi t}}\,(a_0-a_1/2t+1\cdot 3a_2/(2t)^2-\cdots)\ ,$$

whence, by (37),

$$h(t)\sim\frac{1}{\sqrt{\pi t}}\,(a_0-a_1/2t+1\cdot 3a_2/(2t)^2-\cdots)\ ,$$

which is simply the Heaviside expansion (35a). We have thus succeeded in reducing the problem of the asymptotic expansion from the operational equation to that of the asymptotic expansion of the definite integral

$$\frac{1}{\sqrt{\pi}}\int_0^t \frac{f(\tau)}{\sqrt{t-\tau}}\,d\tau\ ,$$

where $f(t)$ is defined and determined by the integral equation

$$F(p) = \int_0^\infty f(t)e^{-pt}dt \ ,$$

If we write the definite integral in the form

$$\frac{1}{\sqrt{\pi t}} \int_0^t d\tau \cdot f(\tau) \cdot (1-\tau/t)^{-1/2} \ , \tag{40}$$

we see at once that the series (39) is obtained by expanding $(1-\tau/t)^{-1/2}$ by the binomial theorem, replacing the upper limit in integration by infinity, and integrating term-by-term.

That the series (39) does represent the behavior of the function (37) for sufficiently large values of t, provided $f(t)$ converges with sufficient rapidity, is evident qualitatively from the form of the definite integral. It can also be shown rigorously that the asymptotic representation of (37) by the series (39) is valid provided the integrals (38) exist. We thus arrive at the conclusions, deducible by rigorous processes, that the validity of the Heaviside rule stated above depends on the properties of the function $f(t)$, defined by the integral equation (36), and in particular on the existence of the infinite integrals (38). On the other hand, the precise sense in which the expansion asymptotically represents the solution will depend on the particular problem, and no general statement on this subject can be made.

We now take up the problem of the expansion solutions, usually divergent, of problems of class 2, formulated by operational equations of the form

$$h = \phi(\sqrt{p}) \ , \tag{41}$$

of which the corresponding integral equation is

$$\frac{1}{p}\phi(\sqrt{p}) = \int_0^\infty h(t)e^{-pt}dt \ . \tag{42}$$

If we suppose that $\phi(\sqrt{p})$ admits of power series expansion in the argument, we may write

$$\phi(\sqrt{p}) = b_0 + b_1\sqrt{p} + b_2 p + b_3 p\sqrt{p} + b_4 p^2 + \cdots, \tag{43}$$

without, however, making any statement regarding the convergence of this expansion.

The Heaviside rule for this type of problem may be stated as follows.

If the operational equation $h = 1/H(p)$ *can be expanded in the form*

$$h = b_0 + b_1\sqrt{p} + b_2 p + b_3 p\sqrt{p} + \cdots,$$

discard all terms containing integral powers of p, *and write*

$$h = b_0 + (b_1 + b_3 p + b_5 p^2 + \cdots)\sqrt{p}\,.$$

The series solution is then obtained by replacing $\sqrt{p}$ *by* $1/\sqrt{\pi t}$ *and* p^n *by* d^n/dt^n, *whence,*

$$h(t) = b_0 + \left(b_1 + b_3\frac{d}{dt} + b_5\frac{d^2}{dt^2} + \cdots\right)\frac{1}{\sqrt{\pi t}} \tag{44}$$

or

$$h(t) = b_0 + \frac{1}{\sqrt{\pi t}}\left(b_1 - \frac{b_3}{2t} + \frac{1\cdot 3 b_5}{(2t)^2} - \frac{1\cdot 3\cdot 5 b_7}{(2t)^3} + \cdots\right). \tag{45}$$

In order to investigate this expansion solution, we construct the auxiliary functions $g, g_1, g_2, \cdots, g_n, \cdots$, defined by the following scheme:

$$g(t) = h(t) - b_0\,,$$

$$g_1(t) = g(t) - \frac{b_1}{\sqrt{\pi t}}\,,$$

$$g_2(t) = t\cdot g_1(t) + \frac{1}{2}\frac{b_3}{\sqrt{\pi t}}\,,$$

$$g_3(t) = t\cdot g_2(t) - \frac{1\cdot 3}{2^2}\frac{b_5}{\sqrt{\pi t}}\,,$$

$$g_4(t) = t\cdot g_3(t) + \frac{1\cdot 3\cdot 5}{2^3}\frac{b_7}{\sqrt{\pi t}}\,,$$

$$\cdots\cdots\cdots\cdots\cdots$$

Successive substitutions in the integral equation (42) and repeated differentiation with respect to p, lead to the set of formulas

$$\int_0^\infty g(t)e^{-pt}dt \sim \frac{b_1}{\sqrt{p}} \quad \text{as} \quad p \to 0 ,$$

$$\int_0^\infty t \cdot g_1(t)e^{-pt}dt \sim -\frac{b_3}{2\sqrt{p}} \quad \text{as} \quad p \to 0 ,$$

$$\int_0^\infty t \cdot g_2(t)e^{-pt}dt \sim \frac{1 \cdot 3}{2^2} \frac{b_5}{\sqrt{p}} \quad \text{as} \quad p \to 0 ,$$

$$\int_0^\infty t \cdot g_3(t)e^{-pt}dt \sim -\frac{1 \cdot 3 \cdot 5}{2^3} \frac{b_7}{\sqrt{p}} \quad \text{as} \quad p \to 0 .$$

Now since

$$\int_0^\infty \frac{e^{-pt}}{\sqrt{\pi t}} dt = \frac{1}{\sqrt{p}},$$

it follows from the preceding that the functions $g_1, g_2, g_3, \cdots$ all converge to zero as $1/t\sqrt{\pi t}$ as $t \to \infty$, provided $g(t)$ *contains no term which converges in an oscillatory manner.* It is this latter restriction, namely that $g(t)$ must contain no oscillatory term, say of the form

$$\frac{1}{\sqrt{\pi t}} \cos t ,$$

which stands in the way of a satisfactory theory.

However, *assuming* that the auxiliary functions $g_1, g_2, g_3, \cdots$ converge to zero as $1/t\sqrt{\pi t}$ as $t \to \infty$, the foregoing leads at once to the formal asymptotic expansion (45). The same procedure, it can be remarked, is applicable to class I problems and gives formally the expansion (35a).

The foregoing cannot be regarded as a satisfactory discussion and is indeed almost as heuristic as Heaviside's own procedure. In addition to the theoretical defects it is un-

satisfactory to the physicist who requires an automatic rule for deriving the asymptotic expansion from the operational equation by algebraic processes without investigation of auxiliary functions, or remainder terms, and in particular an estimate of the numerical error committed by stopping with any term of the series.* Some of these defects it is believed can be overcome; some are inherent, however, in the nature of the problem. For example, in some problems the series is asymptotic in the sense that the error is less than the last term retained; in others it is meaningless for a certain range of values of t, while in yet others, remarkably enough, the series is absolutely convergent for all finite values of t.

We shall now close this paper with a few examples of the series expansion just discussed. The first example will be the asymptotic solution of the operational equation

$$h=\frac{\sqrt{p}}{\sqrt{p+2\rho}}, \tag{46}$$

where ρ is a positive real parameter. The corresponding integral equation is

$$\frac{1}{\sqrt{p^2+2\rho p}}=\int_0^\infty h(t)e^{-pt}dt\,. \tag{47}$$

The physical problem which this represents is the current entering a transmission line of distributed constants L, R, C, in response to a unit e.m.f. applied at time $t=0$.

The solution of equation (47) is known. It is

$$h(t)=e^{-\rho t}I_0(\rho t)\,, \tag{48}$$

where I_0 is the modified Bessel function of the first kind and order zero.

Now return to the operational equation (46). It will be observed that it is of the form $h=F(p)\sqrt{p}$, and hence falls

* Heaviside appears to have regarded the series as asymptotic in the sense that the numerical error is less than the value of the last term retained. This, however, is certainly not true in general.

within class I. Expanding (46) in accordance with the Heaviside rule, we find

$$h=\left\{1-\frac{p}{4\rho}+\frac{1\cdot 3}{2!}\left(\frac{p}{4\rho}\right)^2-\frac{1.3.5}{3!}\left(\frac{p}{4\rho}\right)^3+\cdots\right\}\sqrt{p/2\rho}\,.$$

Now apply the Heaviside rule given above for converting this operational expansion into the explicit solution. We get

$$h(t)\sim\frac{1}{\sqrt{2\pi\rho t}}\left\{1+\frac{1}{8\rho t}+\frac{(1\cdot 3)^2}{2!(8\rho t)^2}+\frac{(1\cdot 3\cdot 5)^2}{3!(8\rho t)^3}+\cdots\right\},$$

which will be recognized as the usual asymptotic expansion of the function $e^{-\rho t}I_0(\rho t)$. The directness of the result, compared with usual methods, is remarkable.

We now take up the example falling in class 2, namely, the operational equation

$$h=e^{-\sqrt{\alpha p}}\,, \tag{49}$$

where α is a positive real parameter. This equation, it may be remarked, formulates the propagated voltage in a non-inductive cable.

The corresponding integral equation is

$$\frac{1}{p}\,e^{-\sqrt{\alpha p}}=\int_0^\infty h(t)e^{-pt}dt\,, \tag{50}$$

of which the solution is known. It is

$$h(t)=\frac{1}{\sqrt{\pi}}\int_0^\tau\frac{e^{-1/\tau}}{\tau\sqrt{\tau}}\,d\tau\,, \tag{51}$$

where $\tau=4t/\alpha$.

Equation (49), it will be observed, falls in class 2; that is, it is of the form $h=\phi(\sqrt{p})$. In accordance with the Heaviside rule for this class, we expand (49) as a power series in $\sqrt{p}$. It is

$$h=1-\frac{\sqrt{\alpha p}}{1!}+\frac{\alpha p}{2!}-\frac{\alpha p\sqrt{\alpha p}}{3!}+\cdots.$$

Rearranging,

$$h=1-\left(1+\frac{\alpha p}{3!}+\frac{(\alpha p)^2}{5!}+\cdots\right)\sqrt{\alpha p}+\left(\frac{\alpha p}{2!}+\frac{(\alpha p)^2}{4!}+\cdots\right).$$

Now apply the Heaviside rule; that is, discard the series in integral powers of p, replace $\sqrt{p}$ by $1/\sqrt{\pi t}$ and p^n by d^n/dt^n in the first bracketed series. We get

$$(52)\quad h(t)=1-\sqrt{\frac{\alpha}{\pi t}}\left(1-\frac{1}{3\cdot 1!}\left(\frac{\alpha}{4t}\right)+\frac{1}{5\cdot 2!}\left(\frac{\alpha}{4t}\right)^2-\cdots\right),$$

which, as it is easy to verify, agrees with (51). The remarkable feature of this example is that the expansion (52) is absolutely convergent, although established by methods applicable to asymptotic expansions.

A final example will be given:

$$(53)\qquad h=\frac{1}{\sqrt{p/\beta}+1},$$

where β is a positive real parameter. This equation formulates a fairly important problem in cable telegraphy, namely the effect of a terminal resistance on the shape of the cable voltage.

The integral equation of the problem is

$$(54)\qquad \frac{1}{p\sqrt{p/\beta}+1}=\int_0^\infty h(t)e^{-pt}dt,$$

of which the solution is known. It is

$$(55)\qquad h(t)=1-e^{\beta t}\sqrt{\frac{\beta}{\pi}}\int_t^\infty \frac{e^{-\beta t}}{\sqrt{t}}\,dt.$$

Equation (53) is of class 2; expanding therefore as a power series in $\sqrt{p}$, we find

$$h=1-\sqrt{p/\beta}+p/\beta-(p/\beta)\sqrt{p/\beta}+\cdots.$$

Applying the Heaviside rule, we get

$$h(t)=1-\frac{1}{\sqrt{\pi\beta t}}\left(1-\frac{1}{2\beta t}+\frac{1.3}{(2\beta t)^2}-\cdots\right). \tag{56}$$

If the definite integral of (55) is asymptotically expanded by repeated partial integrations the resulting series is identical with (56). Furthermore, it is easy to show that it is asymptotic in the sense that the numerical error is less than the value of the last term included.

Borel has employed infinite integrals of the type

$$\int_0^\infty f(t)e^{-pt}dt$$

to sum divergent series. The foregoing suggests that they may be profitably employed to obtain asymptotic expansions of $f(t)$, when such an expansion in inverse fractional powers of t exists.

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NOTE ON A FUNDAMENTAL LEMMA CONCERNING THE LIMIT OF A SUM*

BY H. J. ETTLINGER

In a recent paper† the author presented a proof of a lemma which is of fundamental importance in the theory of the Riemann integral. It is the purpose of this note to replace the proof of Case I of the above paper by the present one which treats the case of unequal subintervals. The theorem by means of which this proof is completed is of interest in itself, and admits of certain generalizations, one of which will be stated in the sequel.

A *horizontal* function of index n, $h(x, n)$, is a function defined on an interval $I \equiv (a \leqq x \leqq b)$ in such a manner that it is possible to divide I into n subintervals $I(i, n)$, $(i = 1, 2, \cdots, n)$, so that $h(x, n) \equiv h_{in} \equiv$ constant, at all interior points of $I(i, n)$. In addition, we define $h(a, n) \equiv h_{1n}$ and $h(b, n) \equiv h_{nn}$. Let $x(i, n)$ be the $n-1$ distinct internal points of division which subdivide I and together with $a \equiv x(0, n)$, $b \equiv x(n, n)$ form the subdivisions, $I(i, n)$, whose lengths are $\Delta_{in} = x(i, n) - x(i-1, n)$. We shall define the value of $h(x, n)$ at the internal points of division $x(i, n)$ to be that value of the pair of numbers h_{in}, $h_{i+1,n}$, whose numerical value is the larger. Furthermore, let

$$f(n) = \sum_{i=1}^{n} h_{in}\, \Delta_{in}\,.$$

The fundamental lemma may now be restated as follows:

Let $h(x, n)$ be a sequence of horizontal functions on I for $n = 1, 2, \cdots$. Let $h(x, n)$ be bounded for all values of n and for

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* Presented to the Society, September 10, 1925.

† *An elementary proof of a fundamental lemma concerning the limit of a sum*, this BULLETIN, vol. 29 (1923), pp. 219–223.

all values of x in I, and let $\lim_{n\to\infty} h(x, n)=0$ *for each fixed x in I. Then* $\lim_{n\to\infty} f(n)=0$.

We shall prove this lemma only for the special case (Case I of the earlier paper) that $|h(x, n+1)| \leqq |h(x, n)|$. From this case, the general case follows in accord with the proof given before. Let

$$\bar{f}(n)=\sum_{i=1}^{n}|h_{in}|\Delta_{in}\,.$$

We have

$$\bar{f}(n)\geqq|f(n)|\geqq 0\,,$$

and

$$\bar{f}(n+1)\leqq\bar{f}(n)\,.$$

We shall show that given any $\epsilon>0$, there exists a positive integer, N_ϵ, such that $|h(x, n)|<\epsilon$, for every x in I and $n\geqq N_\epsilon$. Then $\bar{f}(n)<\epsilon(b-a)$ for $n\geqq N_\epsilon$. Hence $\bar{f}(n)$, and also $f(n)$, approach zero as a limit, as n increases without limit.

THEOREM. *Given any $\epsilon>0$, there exists an N_ϵ such that $|h(x, n)|<\epsilon$ for every x in I and $n\geqq N_\epsilon$.* Suppose the theorem false for the interval I. Let $L(i, n)$ be the subintervals into which I is divided by adjacent pairs of points $x(i, j)$, $i=1, 2, \cdots, j-1$, $j=1, 2, \cdots, n$. If the theorem is false for the interval I, it must be false for at least one of the subintervals $L(1, 2)$, $L(2, 2)$. Let $L(i_2, 2)$ be the first left hand one of these. If the theorem is false for $L(i_2, 2)$, it is false for at least one of the subintervals $L(i, 3)$ of $L(i_2, 2)$. Call the first left hand one of these $L(i_3, 3)$. Continuing this process, we obtain a monotonic non-increasing sequence of intervals, $L(i_n, n)$, which have therefore either a common interval $\bar{L}$ or at least a common point, $\bar{x}$. If $\bar{x}$ be regarded as representing either the common point, or one of the points of $\bar{L}$, all of which have a common $h(\bar{x}, n)$, then since $h(\bar{x}, n)$ approaches zero as a limit as n increases indefinitely, $|h(\bar{x}, n)|<\epsilon$ for n greater than a certain positive integer N_ϵ. This contradicts the hypo-

thesis that the theorem is false for I, and establishes the desired result.

The above theorem may be stated otherwise as follows. If the sequence of horizontal functions $h(x, n)$ satisfying the conditions of the lemma approach the zero limit function monotonically, they must approach it uniformly. It follows as a corollary from this theorem that if we have a series of "interval" functions, such that $f(x, n)$ is a single-valued continuous curve in each subinterval, and if the sequence of functions approach a continuous limit function monotonically for each fixed x in I, then the approach to the limit function is uniform with respect to x. In particular, the sequence of "interval" functions may be a set of functions $f_n(x)$ each of which is continuous in I.

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NOTE ON RATIONAL PLANE CUBICS*

BY C. A. NELSON

1. *Introduction.* Many constructions have been devised for a rational plane cubic. One of the most interesting of them is due to Zahradnik† who noticed that the familiar construction for the cissoid of Diocles could be extended so as to generate any rational plane cubic. It is as follows: Take any conic C, a fixed point O on C, and a fixed line b. Any line l through O meets C a second time at P, and b at Q. On l lay off a segment OM equal to and in the sense PQ. The locus of the point M is a rational plane cubic R with double point at O. The tangents to R and O are the joins of O to the two points in which b meets C.

Niewenglowski‡ showed a bit later that this same construction may be applied to R, using a second fixed line dis-

* Presented to the Society, September 10, 1925.

† Zahradnik, *Cissoidalcurven*, ARCHIV DER MATHEMATIK UND PHYSIK, vol. 56, p. 8.

‡ Niewenglowski, *Sur les courbes d'ordre n à point multiple d'ordre n−1*, COMPTES RENDUS, vol. 80 (1875), p. 1067.

tinct from b. A rational quartic results which has a triple point at O, the triple tangents being the joins of O to the three points in which the second fixed line meets R. Indeed, any rational plane curve of order n having a multiple point of order $n-1$ may be obtained in this way.

Evidently the point M is given by means of a parabolic substitution operating on the line l. The self-corresponding point of this substitution is the point at infinity on l. In this note we introduce homogeneous coördinates which has the effect of replacing the above metric determination of M by a purely projective one. A simple geometric characterization of the relations between the conic C, the line b, and the cubic R is obtained.

2. *Zahradnik's Formula.* Let the fixed point O be (0,0,1) Then the conic C may be written

$$ax^2 + bxy + cy^2 + dxz + eyz = 0 , \tag{1}$$

and the fixed line b

$$px + qy + rz = 0 . \tag{2}$$

The equation of R is, according to Zahradnik,

$$(ax^2 + bxy + cy^2)\,(px + qy + rz) - (dx + ey)\,(px + qy)z = 0 \tag{3}$$

a result which may be readily verified.

The tangent to C at O meets the fixed line b in a point which is a flex for R. The flex-tangent at this point is the line b. We shall not insist upon the verification of these statements at this time as their truth will appear a bit later. Moreover, one would suspect this from the construction.

In order to obtain the desired relations between C and R we solve the converse problem of finding C and b when R is given.

3. *Determination of C and b when R has Distinct Tangents at O.* If R has a double point with distinct—real or imagin-

ary—tangents, we may canonize its equation by means of a convenient choice of the triangle of reference. Let the double point be (0, 0, 1); $y=0$ be a line joining the double point to a real flex-point (there is at least one); $z=0$ be the flex-line (the line on which the three flexes of R lie); $x=0$ be the harmonic of $y=0$ with respect to the pair of tangents to R at O. With these assumptions the equation of R may be written

$$(R) \qquad z(x^2+\epsilon y^2)+y(3x^2-\epsilon y^2)=0\ ,$$

in which ϵ is a real number not zero.

We introduce a parameter λ into this equation by writing it in the form

$$(R_\lambda) \qquad z(x^2+\epsilon y^2)+\lambda y(3x^2-\epsilon y^2)=0\ .$$

The hessian of R_λ is

$$(H_\lambda) \qquad z(x^2+\epsilon y^2)-3\lambda y(3x^2-\epsilon y^2)=0,$$

from which we infer that it also belongs to the family. Hence the curves R_λ form a *syzygetic* family all of whose members have the same double point and tangents there. All the cubics R_λ have the same flex-points $(\sqrt{\epsilon}, \sqrt{3}, 0), (-\sqrt{\epsilon}, \sqrt{3}, 0)$, (1, 0, 0) which we denote by F_1, F_2, F_3, respectively. Consequently, the members of the family have the same flex-line $z=0$. The only points of intersection of different members of the family are the flex-points and the double point. Hence a particular cubic R_{λ_0} may be individualized by specifying a point, other than the above fixed points of intersection, through which it must pass.

For R_λ Zahradnik's formula (3) becomes

$$(4)\quad z(x^2+\epsilon y^2)+\lambda y(3x^2-\epsilon y^2)\equiv(ax^2+bxy+cy^2)(px+qy+rz) -(dx+ey)(px+qy)z\ .$$

There are six essential constants in the equation of a cubic with a double point at a given point, and an equal number in the equations of C and b. Hence, upon equating coefficients

of like terms in (4), a finite number of solutions are obtained. They are

$$(C_\lambda^{(i)}) \qquad \frac{3\lambda}{p_i}xy+\frac{2\epsilon}{3}y^2-\frac{1}{p_i}xz+\frac{2\epsilon}{9\lambda}yz=0\,,$$

$$(b_\lambda^{(i)}) \qquad p_i x-\frac{3\lambda}{2}y+z=0\,, \qquad (i=1,\ 2)$$

where $4\epsilon p^2=27\lambda^2$, the positive root being indicated by p_1;

$$(C_\lambda^{(i)}) \qquad x^2-\frac{\epsilon}{3}y^2-\frac{4\epsilon}{9\lambda}yz=0\,,$$

$$(b_\lambda^{(i)}) \qquad 3\lambda y+z=0\,.$$

4. *Properties of the Solutions.* Denote the point of intersection of $b_\lambda^{(1)}$, $b_\lambda^{(2)}$, by $B_\lambda^{(3)}$, etc. Then

$$B_\lambda^{(1)}=(\sqrt{3\epsilon},\ -1,\ 3\lambda),\quad B_\lambda^{(2)}=(-\sqrt{3\epsilon},\ -1,\ 3\lambda),$$
$$B_\lambda^{(3)}=(0,\ 2,\ 3\lambda)\,.$$

As λ varies $B_\lambda^{(1)}$, $B_\lambda^{(2)}$, $B_\lambda^{(3)}$ move along $x+\sqrt{3\epsilon}\,y=0$, $x-\sqrt{3\epsilon}y=0$, $x=0$, respectively. The first two lines are harmonic with respect to the pair $xy=0$. The lines $b_\lambda^{(1)}$, $b_\lambda^{(2)}$, $b_\lambda^{(3)}$ are the flex-tangents at F_1, F_2, F_3, respectively. The three conics pass through the double point O. They touch by pairs in the three flex-points. In fact, $C_\lambda^{(1)}$ and $C_\lambda^{(2)}$ are tangent at $F_3=(1,\ 0,\ 0)$. Their fourth point of intersection is $(0,-1,\ 3\lambda)$. Similarly, $C_\lambda^{(1)}$ and $C_\lambda^{(3)}$ touch at $F_2=(-\sqrt{\epsilon},\ \sqrt{3},\ 0)$ and have their fourth meet at $(\sqrt{3\epsilon},\ 1,\ 6\lambda)$. The meets of $C_\lambda^{(2)}$ and $C_\lambda^{(3)}$ are found by changing the sign of the radical involving ϵ. Hence the variable point of intersection of the conics, in the above order, always lies on the lines

$$x=0\,,\quad x-\sqrt{3\epsilon}\,y=0\,,\quad x+\sqrt{3\epsilon}\,y=0\,,$$

which are the same as those on which $B_\lambda^{(i)}$ move.

It is a simple matter to show that the polar of $B_\lambda^{(i)}$ with respect to $C_\lambda^{(i)}$ is the line $b_\lambda^{(i)}$ and that the tangents to $C_\lambda^{(i)}$ at

$(0, 0, 1)$ meet R_λ in F_i. Also, that the real points of intersection of $C_\lambda^{(i)}$ and R_λ, other than O, lie on lines which pass through F_i.

The jacobian of the three conics $C_\lambda^{(i)}$ is

$$(J_\lambda) \qquad z(x^2+\epsilon y^2)+3\lambda y(3x^2-\epsilon y^2)=0\ ,$$

so that it also belongs to the syzygetic family. Let (x, y, z) be any point P on J_λ. The three polar lines of P with respect to $C_\lambda^{(i)}$ meet in a point Q whose coördinates are $(\epsilon y^2, -xy, 6\lambda xy+xz)$. This point has been called the *conjugate* of P and lies on J_λ. The conjugates of the three flex-points of the syzygetic family are $Q_1=(\sqrt{3\epsilon}, -1, 6\lambda)$, $Q_2=(-\sqrt{3\epsilon}, -1, 6\lambda)$, $Q_3=(0, 1, 3\lambda)$ from which we infer that the three points Q coincide with the variable points of intersection of the conics $C_{-\lambda}^{(i)}$ belonging to $R_{-\lambda}$. The associated cubics R_λ and $R_{-\lambda}$ meet $x=0$ in two points which form an harmonic pair with $(0, 0, 1)$ and $(0, 1, 0)$. They are the only cubics of the family which meet $x=0$ in these two points.

The polar conic of $R_{-\lambda}$ with respect to any point (α, β, γ) is

$$(5)\quad (\gamma-3\beta\lambda)x^2-6\alpha\lambda xy+\epsilon(\gamma+3\beta\lambda)y^2+2\alpha xz+2\beta\epsilon yz=0\ .$$

For $(\alpha, \beta, \gamma)=(\sqrt{3\epsilon}, -1, 3\lambda)$, $(+\sqrt{3\epsilon}, +1, 3\lambda)$, $(0, 2, -3\lambda)$, the equation (4) gives the three conics $C_\lambda^{(i)}$. But these points are associated with $R_{-\lambda}$ in the same manner that $B_{-\lambda}^{(i)}$ are related to R_λ, that is, they are the points $B_{-\lambda}^{(i)}$. It is a simple matter to show that $B_{-\lambda}^{(i)}$ are the poles of the flex-line $z=0$ with respect to the conics $C_\lambda^{(i)}$.

We may formulate our principal results as follows. *Any rational plane cubic R having distinct tangents at its double point can be generated, in the sense of Zahradnik, in three ways. The three fixed lines are the flex-tangents of the cubic, while the three conics are the polar conics, with respect to $R_{-\lambda}$, of the points of intersection of the three flex-tangents of $R_{-\lambda}$.*

5. *Determination of C and b when R has a Cusp at O.* Let the cusp be at $(0, 0, 1)$ with $x=0$ as the tangent and $(1, 0, 0)$ be the one flex. The equation of R may be written

$$x^2(\alpha y+\beta z)+\gamma y^3=0\ .$$

A real projective transformation—of no geometrical significance—may be found which will give the equation the form

$$x^2(ky+z)+y^3=0\ ,$$

where k may be any real number including zero. For this form $ky+z=0$ is, of course, the flex-tangent.

With the use of formula (3) we again find three solutions for C and b, namely,

$$(C^{(i)}) \qquad 2k^2xy \mp 2k\sqrt{-k}y^2-2kxz \pm \sqrt{-k}yz=0\ ,$$

$$(b^{(i)}) \qquad \pm 2k\sqrt{-k}x-2ky+z=0\ , \qquad (i=1,2),$$

in which $i=1$ refers to the upper sign;

$$(C^{(3)}) \qquad k^2x^2+ky^2+yz=0\ ,$$

$$(b^{(3)}) \qquad ky+z=0\ .$$

Since k may be made to assume any real value without affecting the geometrical significance of our equations, we disregard the first two solutions as giving analytic solutions of no real interest.

As before, the fixed line $b^{(3)}$ is the flex-tangent. A simple calculation shows that $C^{(3)}$ meets the cubic R in only two real points, other than (0, 0, 1). These points lie on the line $2ky+z=0$. The two pairs of lines $y=0$, $2ky+z=0$ and $ky+z=0$, $z=0$ are harmonic. Moreover, the two tangents to $C^{(3)}$ from the flex-point are the flex-tangent and $y=0$. These properties serve to characterize $C^{(3)}$ and $b^{(3)}$. In fact, if we are given any cuspidal cubic referred to a triangle whose sides are the cuspidal tangent, the line joining the cusp to the flex-point, and a third line through the flex-point which is not the flex-tangent, we obtain the line b by drawing the flex-tangent. The conic C is determined as that conic whose tangents from the flex-point are the line joining the double point to the flex, and the flex-tangent; and which meets R in the two points cut out from R by the line through the flex harmonic with $y=0$ with respect to the pair $z=0$ and the flex-tangent. These conditions completely determine C.

JOHNS HOPKINS UNIVERSITY

A TRIVIAL TAUBERIAN THEOREM.*

BY W. A. HURWITZ

The name Tauberian was introduced by Hardy† to describe a very interesting type of theorem in connection with summable series; he and others have enunciated a considerable number of such theorems bearing on various specific definitions of summability. We may indicate the general character of a Tauberian theorem as follows. The ordinary questions on summability consider two related sequences (or other functions) and ask whether it will be true that one sequence possesses a limit whenever the other possesses a limit, the limits being the same; a Tauberian theorem appears, on the other hand, only if this is untrue, and then asserts that the one sequence possesses a limit provided the other sequence both possesses a limit and satisfies some additional condition restricting its rate of increase. The interest of a Tauberian theorem lies particularly in the character of this additional condition, which takes different forms in different cases. Thus, if the condition is to be imposed on the term u_n of a series, it may take any of the forms (for which I write alternative notations)

$$|u_n| < Kf_n : \qquad u_n = O(f_n) ,$$

$$u_n < Kf_n : \qquad u_n = O_+(f_n) ,$$

$$u_n > -Kf_n : \qquad u_n = O_-(f_n) ,$$

$$u_n/f_n \to 0 : \qquad u_n = o(f_n) ,$$

$$\limsup_{n=\infty} u_n/f_n = 0 : \quad u_n = o_+(f_n) ,$$

$$\liminf_{n=\infty} u_n/f_n = 0 : \quad u_n = o_-(f_n) ;$$

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* Presented to the Society, December 30, 1924.

† With respect to the name, see Hardy and Littlewood, PROCEEDINGS OF THE LONDON SOCIETY, vol. 2, (1912-13), p. 1.

in each case K is a positive constant and (f_n) a given sequence of positive elements.

It has seemed to me worth while to discuss a case so trivial that the reasoning is transparent, but involving a parameter in such a way that for various values of the parameter it exhibits several forms of the condition, with assurance that the best form for each value has been chosen.*

Consider then the series

$$u_1+u_2+u_3+\cdots;$$

let

$$x_n=u_1+u_2+\cdots+u_n.$$

Apply the transformation

$$(A) \qquad y_1=\frac{x_1}{1-\alpha}; \qquad y_n=\frac{x_n-\alpha x_{n-1}}{1-\alpha}, \qquad n>1,$$

where α is any real number other than 1. It is at once obvious that (A) is regular: whenever x_n has a limit, y_n has the same limit.†

We now inquire whether the existence of a limit for y_n implies the existence of a limit for x_n. For the inverse transformation we find

$$(A^{-1}) \quad x_n=(1-\alpha)\left[y_n+\alpha y_{n-1}+\alpha^2 y_{n-2}+\cdots+\alpha^{n-1}y_1\right].$$

This is of the form $x_n=\sum a_{n,k}y_k$, with

$$a_{n,k}=(1-\alpha)\alpha^{n-k},\ k\leqq n;\ a_{n,k}=0,\ k>n.$$

Thus, by the Silverman-Toeplitz Theorem,‡ if A^{-1} is to be regular, $|\alpha|<1$; and if $|\alpha|<1$, since

$$\sum_{k=1}^{n}a_{n,k}=1-\alpha^n\to 1\ ;\ \sum_{k=1}^{n}|a_{n,k}|=\frac{|1-\alpha|}{1-|\alpha|}\left[1-|\alpha|^n\right]<\frac{|1-\alpha|}{1-|\alpha|},$$

* A recent paper by R. Schmidt, MATHEMATISCHE ZEITSCHRIFT, vol. 22 (1925), pp. 89–152, for the first time undertakes a systematic general study of Tauberian theorems. Schmidt's work yields as special cases some, but not all, of the results of this note.

† This holds also if α is complex.

‡ See the author's *Report on topics in the theory of divergent series*, this BULLETIN, vol. 28 (1922), p. 19.

by the same theorem, A^{-1} will be regular. In case, then, $|\alpha|<1$, A is equivalent to convergence; no Tauberian condition is required to insure that if y_n has a limit, x_n shall have the same limit.

Suppose now $|\alpha|>1$. Write

$$\lambda=(1-\alpha)\left[\frac{y_1}{\alpha}+\frac{y_2}{\alpha^2}+\cdots\right];$$

the series converges, since y_n is bounded. We find

$$x_n-\lambda\alpha^n=(\alpha-1)\left[\frac{y_{n+1}}{\alpha}+\frac{y_{n+2}}{\alpha^2}+\cdots\right].$$

The right-hand side of this equation is in the form $\sum b_{n,k}\,y_k$, where

$$b_{n,k}=0\,,\quad k\leqq n\,;\quad b_{n,k}=\frac{\alpha-1}{\alpha^{k-n}}\,,\quad k>n\,.$$

Hence

$$\lim_{n\to\infty} b_{n,k}=0\,;\quad \sum_{k=1}^{\infty}b_{n,k}=1\,;\quad \sum_{k=1}^{\infty}|b_{n,k}|=\frac{|\alpha-1|}{|\alpha|-1}\,;$$

and by the Hildebrandt-Carmichael generalization* of the Silverman-Toeplitz theorem, the transformation $||b_{n,k}||$ is regular. Since $y_n\to l$,

$$x_n-\lambda\alpha^n\to l\,,$$

and

$$u_n-\lambda\alpha^{n-1}(\alpha-1)\to 0\,.$$

We now consider separately the cases $\alpha>1$, $\alpha<-1$.

Let $\alpha>1$. Then if $\lambda\neq 0$, $x_n\to(\operatorname{sgn}\lambda)\infty$, $u_n\to(\operatorname{sgn}\lambda)\infty$; if $\lambda=0$, $x_n\to l$, $u_n\to 0$. In order to insure by a condition on u_n that $x_n\to l$ we must prevent u_n from becoming definitely (i. e., with one sign) infinite. This is secured by demanding that u_n be bounded *both above and below*:

$$|u_n|<K:\quad u_n=O(1)\,.$$

* See the author's *Report*, loc. cit., p. 20; also Hildebrandt, this BULLETIN, vol. 29 (1923), p. 314.

Let $\alpha<-1$. Then if $\lambda\neq 0$, x_n and u_n each oscillate between $+\infty$ and $-\infty$; if $\lambda=0$, $x_n\to l$, $u_n\to 0$. In order to insure that $x_n\to l$, it suffices to bound u_n *in one direction only*; that is, to impose *one* of the conditions

$$u_n<K: \qquad u_n=O_+(1);$$

$$u_n>-K: \qquad u_n=O_-(1).$$

There remains the case $\alpha=-1$. Here

$$y_n=\frac{x_n+x_{n-1}}{2}, \qquad n>1. \tag{1}$$

Neither of the preceding methods applies. The failure, furthermore, lies not merely in the proof, but in the facts. Conditions of the forms O, O_+, O_- are not sufficient; it is possible for y_n to possess a limit while x_n and u_n oscillate finitely or infinitely. For instance, let

$$u_1=l+a: \quad u_n=(-1)^{n-1}2a, \quad n>1;$$

where $a>0$. Then

$$x_n=l+(-1)^{n-1}a;$$

$$y_1=\frac{l+a}{2}; \quad y_n=l, \quad n>1.$$

If we call lim sup $x_n=\bar{x}$, lim inf $x_n=\underline{x}$ (whether finite or infinite), with similar notations for u_n,

$$y_n\to l; \quad \bar{x}=l+a, \quad \underline{x}=l-a; \quad \bar{u}=2a, \quad \underline{u}=-2a.$$

Again, let

$$y_n=\frac{(-1)^{n-1}}{n};$$

then

$$x_n=(-1)^{n-1}2\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right);$$

$$u_n(=-1)^{n-1}\left[-\frac{2}{n}+4\left(1+\frac{1}{2}+\cdots+\frac{1}{n}\right)\right];$$

so that

$$y_n \to 0 \; ; \; \bar{x} = +\infty \; ; \; \underline{x} = -\infty \; ; \; \bar{u} = +\infty \; , \; \underline{u} = -\infty \; .$$

To deduce an appropriate condition in this case, note that

(2) $$y_n = x_n - \tfrac{1}{2}u_n \; ; \quad y_n = x_{n-1} + \tfrac{1}{2}u_n \; , \quad n > 1 \; .$$

From (1) and (2), we have

$$\bar{x} = 2l - \underline{x} \; ;$$

$$\bar{x} = l + \tfrac{1}{2}\bar{u} \; , \quad \bar{x} = l - \tfrac{1}{2}\underline{u} \; ;$$

$$\underline{x} = l + \tfrac{1}{2}\underline{u} \; , \quad \underline{x} = l - \tfrac{1}{2}\bar{u} \; .$$

Thus if any one of the four limits $\bar{x}$, $\underline{x}$, $\bar{u}$, $\underline{u}$ is finite, all are finite and $\bar{u} = -\underline{u}$. If none are finite, then $\bar{x} = \bar{u} = +\infty$, $\underline{x} = \underline{u} = -\infty$. In order to be able to assert that $x_n \to l$, we must have $\bar{x} = \underline{x} = l$, hence $\bar{u} = \underline{u} = 0$. But if *either* $\bar{u} = 0$ or $\underline{u} = 0$, it follows that $\bar{x} = l$, $\underline{x} = l$, and hence $x_n \to l$. Therefore it suffices in this case to impose one of the conditions

$$\limsup_{n \to \infty} u_n = 0 : \quad u_n = o_+(1) \; ;$$

$$\liminf_{n \to \infty} u_n = 0 : \quad u_n = o_-(1) \; .$$

The results may be collected as follows:

For the transformation A, if y_n has the limit l, x_n will also have the limit l:

in case $\alpha < -1$, if $u_n = O_+(1)$ or $u_n = O_-(1)$;

in case $\alpha = -1$, if $u_n = o_+(1)$ or $u_n = o_-(1)$;

in case $-1 < \alpha < 1$, without any Tauberian condition;

in case $\alpha > 1$, if $u_n = O(1)$.

Thus the cases of bilateral O-condition, unilateral O-condition, and absence of Tauberian condition occur for whole intervals of parameter values, separated by isolated points which are characterized by o-conditions or by the breaking down of the transformation. It is also of interest that for $\alpha > 1$, the form of condition given may be replaced by the formally weaker condition,

$$\liminf_{n \to \infty} u_n \neq +\infty \quad \textit{and} \quad \limsup_{n \to \infty} u_n \neq -\infty \; ;$$

and for $\alpha = -1$, the condition given may be replaced by

$$\lim_{n\to\infty} u_n \text{ exists.}$$

Extensions can be made by allowing y_n to involve more than two consecutive x's. If for example

$$y_n = \frac{x_n + \alpha x_{n-1} + \beta x_{n-2}}{1+\alpha+\beta},$$

the different conditions depend on the location of the roots (real or complex) of the polynomial $z^2+\alpha z+\beta$ with respect to the unit circle $|z|=1$. No new kinds of result appear, and the analysis is less transparent; as I consider the Tauberian theorem of this note to have no intrinsic importance and to be of interest merely as an indication of general relationship of Tauberian conditions to one another, I omit the discussion of the extensions mentioned.

CORNELL UNIVERSITY

COOLIDGE ON PROBABILITY

An Introduction to Mathematical Probability. By Julian L. Coolidge. Oxford, Clarendon Press, 1925. xi+215 pp.

This book meets a real need for a textbook on probability written in English and giving an introduction to a wide range of topics. The work is based on lectures by Professor Coolidge given at Harvard University. The purpose expressed in the preface is to give the mathematical basis underlying each of the important applications of probability rather than to write a treatise on games of chance, or errors of observations, or statistics, or statistical mechanics, or insurance. In the opinion of the reviewer, the author has succeeded well in carrying out his purpose in brief space.

The first chapter deals with the meaning of probability. Various definitions of probability are described. Then by means of three broad empirical assumptions the statistical definition of probability is given and adhered to throughout the book. On the whole, a good case seems to be made for the underlying rationale adopted. Moreover, the definition adopted surely lends itself to important statistical applications. The elementary principles for the combination of probabilities and the meanings of expectation and risk are clearly developed in the second chapter. The third chapter deals with the Bernoulli theorem which is rather commonly regarded as the central theorem underlying the main applications of probability in practical statistics. A very commendable feature of this chapter is the clearness and emphasis with which approximate results are set down as approximations. For example, it is emphasized that the Gaussian probability integral is merely an approximation to the probability, in repeated trials, that the discrepancy is numerically equal to or less than an assigned value. Hence, it is properly called a "fortunate accident" that

$$\frac{2}{\pi}\int_0^\infty e^{-x^2}\,dx=1\,,$$

and it is another fortunate accident that

$$\frac{1}{\sqrt{2\pi npq}}\int_0^\infty x^2 e^{-\frac{x^2}{2npq}}\,dx=npq\,.$$

An unfortunate slip occurs on page 36. By simply taking the sense of the inequality $n-r\leqq r$ to be $n-r\geqq r$, certain incorrect conclusions are drawn. These errors are to be corrected at once by an erratum slip. The chapter on mean values and dispersion seems to the reviewer to be one of the most interesting chapters of the book, particularly because it presents the author's extension of the Lexis theory to mean values of observations

instead of restricting the theory to mean values of relative frequencies. The chapter on probability of causes is a brief treatment along fairly conventional lines. Bayes' principle is regarded as open to grave question. Certain necessary conditions and limitations on the use of the principle are well discussed on page 100. No attention seems to be given to the claim of Karl Pearson for his recent improved basis for the probability of causes published in BIOMETRIKA, (vol. 13, 1920–21, p. 1). After considering objections to the Bayes' formula a question is asked and answered as follows: "Why not, then, reject the formula outright? Because, defective as it is, Bayes' formula is the only thing we have to answer certain important questions which do arise in the calculus of probability." In this connection the reader will surely enjoy the human touch in the following remark: "Therefore we use Bayes' formula with a sigh, as the only thing available under the circumstances: 'Steyning tuk him for the reason the thief tuk the hot stove—bekaze there was nothing else that season.' " The two chapters on the distribution and treatment of errors of observations are of more than ordinary interest. The feature of these chapters which appeals to the reviewer particularly is the explicit statement of the underlying assumptions. The derivation of the law of errors in many variables is taken from the author's article *The Gaussian law of errors for any number of variables*, TRANSACTIONS OF THIS SOCIETY, 1923. The idea of correlation comes up in connection with this generalized theory. Correlation is thus approached from the frequency surface standpoint. Later the regression equation standpoint is dealt with briefly. We are told (p. 149) that "The great trouble at present with the theory of correlations seems to be that there is no general agreement as to how large r^2 must be in order so that we may safely conclude there is a real connection between the two sets of phenomena." In the opinion of the reviewer this trouble may be largely overcome in statistical practice if we keep clearly before us the purpose for which we are to use the correlation coefficient. For example, if the purpose is merely to establish the existence of correlation, then a very small value of r may indicate a connection. In this case, the comparison of the correlation coefficient should be with multiples of its probable error. On the other hand, if the purpose is to predict the value of a variable y corresponding to an assigned x, it is not likely that a useful prediction can be made unless r is relatively large. Between these two extremes, we may find a moderate value of r useful in predicting the average values in the arrays of y's corresponding to assigned x's. Thus, the trouble mentioned is likely to be overcome in part at least when the purpose is clearly defined. The brief chapter on statistical theory of gases will serve the very useful purpose of introducing an important application of probability to a wider group of mathematical readers. In the chapter on principles of life insurance the difficulties of obtaining probabilities of life and death from statistical data are clearly discussed. The statement (p. 194) that "A life table calculated by Makeham's formula is better than any but the best table calculated by other means" is likely to be misleading. The Makeham formula gives a useful approximation to certain tables. For other well prepared tables such as the recent American-

Canadian Men Table the approximation is not satisfactory for many purposes.

The following minor typographical errors should be noted:

Page 46, formula (11), $\frac{1}{\sqrt{2npq}}$ should read $\frac{1}{\sqrt{2\pi npq}}$.

Page 48, formula (14), the $\sqrt{2npq}$ should read $\sqrt{2\pi npq}$.

Page 154, lines 19 and 24, the words "normal" and "residual" should be interchanged.

Page 170, line 6, $s_m = \sum_{k=0}^{k=m} x_k y_k$ should read $\frac{1}{m!} \sum_{k=0}^{k=n} x y_k^m$.

Then formula (36) following should begin with a factor $m!$

The author directed my attention to the last two errors noted.

In conclusion, the space available permits the reviewer to add only one or two instances of the very fine human touch that gives much life to the exposition. For example, on page 170 we note reference to "this unlovely formula." In addition to certain classical problems on probability involving a human element, we are pleased with such new ones as the following (p. 25) related to De Montmort's example and ascribed to one of the author's colleagues: "If all the inhabitants of Chicago should meet together in one place and get extremely drunk, and then try to go home by guesswork, the chances that at least one would get back to his own bed is almost two out of three." Then it is aptly remarked that "this is one of those cases where it is fortunate that the probability can be calculated beforehand, and we are not forced to seek it experimentally."

H. L. RIETZ

SHORTER NOTICES

Bibliographie des Triangles Spéciaux. By J. Neuberg. Brussels, 1924. 52 pp.

In this bibliography some 50 different special triangles are listed and their curious properties, in so far as they concern the modern geometry of the triangle, are enumerated with explicit references to the literature. This is often accompanied by systematic developments and proofs, and with additions by the compiler. Of its usefulness there is certainly no question.

As to completeness, probably no bibliography of any field in elementary geometry will ever be complete, but this particular field has long enjoyed a certain advantage. It must be remembered that for over forty years Neuberg has been a worker in and bibliographer of modern geometry. Over this tiny corner of the great domain he seems to have watched and labored with unflagging interest and fully one-half of the 170 references listed are to MATHESIS of which he is an editor. This last statement which might at first sight argue against completeness, in this case means definitely that investigations in this field have been drawn to this journal. We note the absence of reference to American journals, and a cursory examination leads us to believe that this reflects in no way on Neuberg. We trust this absence will continue, a statement we hasten to explain. The student who has at his disposal MATHESIS and this bibliography has nearly all the information on this subject that he needs. This concentration should be maintained, and teachers conducting courses in modern geometry should urge their pupils to survey this charming little field, and to communicate new results to MATHESIS.

B. H. BROWN

The Law of Diminishing Returns. By W. J. Spillman. Yonkers, World Book Co., 1924. xi+178 pp.

This little book is very interesting, but more particularly to workers in biology and agriculture. It consists of two parts: The Law of the Diminishing Increment and The Law of the Soil. The first part deals with a special phase of the law of diminishing returns so familiar to students of economics and refers to the more precisely defined law which states that the increments in yield corresponding to successive equal increments in fertilizer applied to a crop tend to constitute the terms of a decreasing geometric series. Mathematicians would be interested primarily in the numerous numerical confirmations of the law by field experiments with fertilizer and with irrigation water; also its application to the growth of fattening animals, including children. The second part is a translation of a discussion of the results of experiments along the same line conducted in Germany by Mitscherlich.

Errors occur among the differences in the table on page 4; at the top of page 10, certain numbers should be written as subscripts.

C. H. FORSYTH

Die Differentialgleichungen des Ingenieurs. By W. Hort. Second edition. Berlin. Julius Springer, 1925. 12+700 pp.

The first edition of this book appeared in 1914 and was reviewed in this BULLETIN, vol. 23, p. 379. Since general conditions have been approximately normal, the first edition has been exhausted with unusual rapidity for a book of such specialized character. Apparently the author has succeeded in filling a need in technical literature.

For the second edition the material has been expanded by a more complete treatment of the methods of numerical approximation for ordinary differential equations and by the addition of a section on the calculus of variations and a section on integral equations.

As the title indicates, the book is written for engineers. The author's point of view is set forth in a quotation from Burkhardt to the effect that the study of technical mathematics should be pursued like the study of a natural science, leaving aside the finer researches and depending upon common sense, technical insight, and experiment to avoid the grosser errors. Accordingly, one finds in this book many technical and physical examples to illustrate the methods of integration, but little consideration of questions regarding the existence of solutions and the convergence of the series employed.

The usual types of ordinary differential equations are taken up and each type is illustrated by a technical application. Considerable space is given to the theory and description of various types of integrating machines, and the best methods of numerical approximation are explained and illustrated. These distinctive features give the book its value to the scientific worker who uses a differential equation as a tool.

The author has not confined himself entirely to technical examples, but has introduced others when the methods employed might be useful in engineering problems. For this reason and because of its general interest a brief study of central forces is followed by a rather complete development of elliptic motion under the newtonian law. A section on linear difference equations is also included.

The study of partial differential equations is confined to the standard equations occurring in mathematical physics. The motion of elastic bodies, problems of hydrodynamics, and an extensive discussion of the potential serve to introduce Bessel's functions and spherical harmonics and to illustrate the methods of finding the solutions.

The usefulness of the book is increased by a very complete index, a list of references to the literature, and a tabulation of the types of equations treated, together with the illustrative examples used under each type.

The mechanical construction of the book is exceptionally good. The type is clear, the figures are carefully made and clearly reproduced, and the quality of paper is very much better than has been commonly used by German printers in recent years.

W. R. LONGLEY

An Introduction to the Operations with Series. By I. J. Schwatt. Philadelphia, Press of the University of Pennsylvania, 1924. 10+287 pp.

According to the preface, "The matter contained in this book had its inception in the author's effort to obtain the value for the sum of the series of powers of natural numbers, in an explicit form and without the use of the Bernoulli numbers. This problem led to the study of the higher derivatives of functions of functions, which in turn required certain principles in operations with series, which had to be established. By means of these and other principles, methods for the expansion of certain functions and the summation of various types of series were devised and other topics developed."

The book starts out with the development of formulas for the nth derivative of various types of functions of functions of algebraic and trigonometric forms, and thus obtains the nth term of the Maclaurin expansion of these functions. Methods for the summation of various types of finite and infinite series are obtained. In the final chapter expressions are derived for the Bernoulli and Euler numbers which the author believes to be simpler and obtained by less laborious methods than those known heretofore.

A great number of formulas are derived, and many special results of some value are obtained. A number of ingenious devices are used. The methods are rather special, and no new general principles are developed. While the problem of determining the nth derivatives or the nth terms of the expansions of various functions is apparently solved by the derivation of definite formulas, yet the results are not entirely satisfactory for practical use as they involve finite summations not easily handled.

L. L. SMAIL

A History of Mathematics in Europe. By J. W. N. Sullivan. London and New York, Oxford University Press, 1925. 109 pp.

This book is not as pretentious as the title implies since it is Chapter 4 in the *History of Science* edited by Charles Singer and covers the period from Boethius (475-526) through Laplace (1749-1827). The subtitle states that it is an account of the progress of mathematics from the fall of Greek science to the conception of rigorous proofs and is based upon the histories of Cantor, Ball, and Cajori. Anyone who wishes to avoid reading these histories can gain a rudimentary knowledge of the subject by reading this book, which is well written, and the reader can find the essential things without too much effort.

The illustrations are taken verbatim from the aforementioned works and although one gets nothing new from this book it should be in every library, especially high school libraries, as it is so easy to find what one wants. More volumes of this character should be welcomed by those who give courses in the history of mathematics.

G. H. LIGHT

NOTES

The concluding number of volume 27 of the TRANSACTIONS OF THIS SOCIETY (October, 1925) contains the following papers: *Cone cubic configurations of a ruled surface*, by A. F. Carpenter; *Concerning upper semi-continuous collections of continua*, by R. L. Moore; *On the oscillation of a continuum at a point*, by W. A. Wilson; *On the conditions of integrability of covariant differential equations*, by J. A. Schouten; *Complete groups of points on a plane cubic curve of genus one*, by M. I. Logsdon; *On the existence of the Stieltjes integral*, by H. L. Smith; *The mutual inductance of two square coils*, by T. H. Gronwall; *On the development of continuous functions in series of Tchebycheff polynomials*, by J. A. Shohat; *Analytic functions in three dimensions*, by E. R. Hedrick and L. Ingold; *The Beltrami equations in three dimensions*, by E. R. Hedrick and L. Ingold; *Fields of parallel vectors in a Riemannian geometry*, by L. P. Eisenhart; *A symbolic treatment of the geometry of hyperspace*, by L. Ingold.

The third number of volume 47 of the AMERICAN JOURNAL OF MATHEMATICS (July, 1925) contains: *A configuration of thirteen pencils of cubics and cubics with three real inflexions*, by B. M. Turner; *A mathematical theory of competition*, by C. F. Roos; *Imprimitive substitution groups*, by G. A. Miller; *Classification of monoidal involutions having a fixed tangent cone*, by M. M. Torrey; *Self-projective rational septimics*, by R. M. Winger.

The concluding number of series 2, volume 26, of the ANNALS OF MATHEMATICS (June, 1925) contains: *On continued fractions in the theory of binary forms*, by A. Arwin; *A property of sequences of Laplace*, by H. L. Olson; *Green's lemma*, by H. E. Bray; *A class of minimum problems and the linear independence of functions of one variable*, by O. Dunkel; *Tables of quadratic forms*, by A. E. Cooper.

At the Southampton meeting of the British Association for the Advancement of Science, Professor Horace Lamb delivered as incoming president his inaugural address, entitled *The figure and constitution of the earth*. The Association will hold its next meeting at Oxford, August 4-11, 1926, under the presidency of the Prince of Wales.

The French Association for the Advancement of Science met at Grenoble, July 27, 1925, under the presidency of Professor Emile Borel, minister of the Navy in Painlevé's cabinet.

At the meeting of the International Research Council at Brussels, Professor Emile Picard was unanimously reelected president. No change in the basis of membership in the Council was made.

At the meeting of the International Astronomical Union, at Cambridge, July 14-22, 1925, Professor W. de Sitter was elected president for the next three years. On this occasion Cambridge University conferred honorary doctorates on Retiring President W. W. Campbell, Dr. B. Baillaud, and Professors de Sitter, Nagaoka, and Schlesinger.

The Russian Academy of Sciences, founded at the end of the reign of Peter the Great, celebrated its two hundredth anniversary September 5-14, 1925.

Mr. Nalini Rajan Choudhury has donated a medal, to be known as the Mahamaya Gold Medal, to be awarded under the auspices of Calcutta Mathematical Society for an essay on the Hindu-Arabic numerals. Competition is restricted to graduates of or students in an Indian university.

The Brussels Academy has awarded its Agathon de Potter prize for the period 1922-24 to Paul Ver Eecke, for his work *Les Coniques d'Apollonius de Perge.*

Cambridge University has awarded its John Winbolt prize jointly to C. S. Deakin, of Queen's College, and W. B. C. Perrycoste, of Sidney Sussex College, for their dissertation entitled *Theory of transverse oscillations of girders.*

Dr. Irving Langmuir, of the General Electric Company, Schenectady, has been awarded the Cannizzaro prize of the Reale Accademia dei Lincei.

The collection of astronomical and mathematical instruments, dating back to the tenth century and illustrating the early history of the sciences, which has been presented by Mr. Lewis Evans to Oxford University and housed in the old Ashmolean Building, was opened for public exhibition on May 5, 1925. On this occasion the honorary degree of doctor of science was conferred on Mr. Evans.

Professor Paul Appell has been appointed honorary rector of the University of Paris, and awarded the grand croix de la Légion d'Honneur.

Professor Henri Lebesgue has been elected a foreign member of the Reale Accademia dei Lincei.

Professor L. Bieberbach, of the University of Berlin, has been elected an honorary member of the Benares Mathematical Society.

Professors Neils Bohr and Max von Laue have been selected foreign corresponding members of the Vienna Academy of Sciences.

The University of Cologne has conferred an honorary doctorate on W. von Crayen, of the scientific publishing firm of W. de Gruyter, of Berlin.

Professor Albert Einstein has been elected foreign associate of the Italian Society of Sciences (the XL).

Professor L. Henneberg has been made an honorary member of the senate of the Darmstadt Technical School.

Trinity College, Dublin, has conferred the honorary degree of doctor of science on Professor R. A. Millikan, of the California Institute of Technology.

Mr. P. P. A. Aiyar, of Judson College, Rangoon, has been appointed lecturer in mathematics at the Government Technical Institute, Insein, Burma.

Dr. Helmut Hasse, of the University of Kiel, has been appointed professor of mathematics at the University of Halle.

Dr. Lothar Koschmieder, of the University of Breslau, has been promoted to an associate professorship.

Associate Professor E. Pohlhausen has been promoted to a full professorship at the University of Rostock.

Professor J. Radon, of the University of Greifswald, has been appointed professor of mathematics at the University of Erlangen.

Associate Professor P. Reidemeister, of the University of Vienna, has been appointed to a full professorship at the University of Königsberg.

Dr. G. Szegö, of the University of Berlin, has been promoted to an associate professorship.

Professor Hermann Weyl, of the Zurich Technical School, has been appointed to a professorship at the University of Leipzig.

The following have been appointed privat docents in mathematics in German universities: Dr. E. Bessel-Hagen, at the University of Göttingen; Dr. H. Geppert, at the University of Giessen; Dr. Robert Schmidt, at the University of Königsberg.

M. J. Kampé de Fériet, maître de conférences, has been promoted to a professorship at the University of Lille.

M. Paul Lévy has been appointed professor of mechanics at the Ecole Supérieure des Mines.

Professor L. Berzolari, of the University of Milan, has returned to his former professorship at the University of Pavia.

At the University of Cagliari, Professor R. Serini has been appointed associate professor of rational mechanics. Professors B. Caldonazzo and O. Chisini have been transferred, Professor Caldonazzo to the chair of rational mechanics at the University of Catania, and Professor Chisini to the chair of analytic and projective geometry at the University of Milan.

Mr. N. M. H. Lightfoot has been appointed assistant lecturer in mathematics at the University of Sheffield.

Mr. E. C. Titchmarsh, of University College, London, has been appointed to the university readership tenable at the College.

At the University of Iowa, Associate Professor E. W. Chittenden has been promoted to a full professorship, and Dr. Roscoe Woods to an assistant professorship of mathematics. Dr. C. C. Wylie, of the University of Illinois, has been appointed assistant professor of astronomy.

At the University of Pennsylvania, Assistant Professor M. J. Babb has been promoted to a full professorship, and Dr. J. D. Eshleman to an assistant professorship of mathematics. Dr. P. A. Caris has been appointed assistant professor.

Assistant Professor A. D. Campbell, of the University of Arkansas, has been promoted to an associate professorship of mathematics.

Dr. H. B. Curtis, of Northwestern University, has been appointed associate professor of mathematics at Marquette University.

Mr. E. E. Erickson, of the University of Iowa, has been appointed to an assistant professorship at Miami University.

Associate Professor Hillel Halperin has been promoted to a full professorship of mathematics at the Agricultural and Mechanical College of Texas.

Miss Frances Harshbarger, of West Virginia University, has been appointed head of the department of mathematics at Potomac State College, Keyser, W. Va.

Dr. D. L. Holl has been appointed assistant professor of mathematics at Iowa State College.

Professor L. M. Hoskins, of the department of applied mathematics at Stanford University, has retired from active teaching.

Miss Dora E. Kearney has been appointed professor of mathematics at Iowa State Teachers College, Cedar Falls.

Dr. Israel Maizlish has been appointed associate professor of physics at Centenary College, Shreveport, La.

Dr. D. H. Menzel, of the University of Iowa, has been appointed assistant professor of astronomy at Ohio State University.

Assistant Professor A. L. Nelson, of the University of Michigan, has been appointed professor of mathematics at the College of the City of Detroit.

Professor R. L. Newlin, of Guilford College, has been appointed assistant professor of mathematics at Ohio Wesleyan University.

Professor W. F. Shenton, of the United States Naval Academy, has been appointed professor of mathematics and physics at the American University, Washington, D. C.

Professor H. L. Smith, of the University of the Philippines, has been appointed to an assistant professorship at the University of Minnesota.

Mr. H. S. Vandiver, of Cornell University, has peen appointed associate professor of mathematics at the University of Texas.

Assistant Professor F. M. Weida, of Montana State College, has been appointed assistant professor of mathematics at Lehigh University.

Professor Gregorio Ricci Curbastro, of the University of Padua, founder of the absolute differential calculus used in the development of relativity theorey, died August 6, 1925, at the age of seventy-two.

Professor J. L. W. V. Jensen, of Copenhagen, died April 5, 1925.

Professor G. Frege, retired, of the University of Jena, died July 26, 1925.

Professor von Hammer of the department of geodesy and practical astronomy of the Stuttgart Technical School, died September 11, 1925, at the age of sixty-seven.

Professor Gordon Armstrong, head of the department of mathematics at Ohio Wesleyan University and secretary-treasurer of the Ohio Section of the Mathematical Association of America, died January 8, 1926.

The death is announced of Professor B. H. Kerstein, of the State Teachers College, Silver City, New Mexico.

Miss Marcia Latham, of Hunter College, died May 9, 1925. Her translation of Descartes' *Geometry*, prepared in collaboration with Professor D. E. Smith, recently appeared through the Open Court.

Assistant Professor J. W. Miller, of the University of Pennsylvania, died September 11, 1925, at the age of forty-nine.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ALTSHILLER-COURT (N.). College geometry. Richmond, Johnson Publishing Company, 1925. 13+250 pp. $4.00

ANDOYER (H.). See HANOTAUX (G.).

APPELL (P.). Sur les fonctions hypergéométriques de plusieurs variables, les polynomes d'Hermite et autres fonctions sphériques dans l'hyperespace. (Mémorial des Sciences Mathématiques, No. 3.) Paris, Gauthier-Villars, 1925.

BIANCHI (L.). Lezioni di geometria differenziale. 3a edizione interamente rifatta. Volume 2, parte 1 e parte 2. Bologna, Zanichelli, 1923-24. 8+412+5+833+11 pp.

BIEBERBACH (L.). Lehrbuch der Funktionentheorie. 2te Auflage. I: Elemente der Funktionentheorie. Berlin, 1923. 312 pp.

BLOCH (W.) und FUHLBERG-HORST (J.). Ein Taschenbuch für Freunde der Mathematik. Band 1. Stuttgart, Franckhsche Verlagshandlung, 1925. 80 pp.

BOSMANS (H.). See STEVIN (S.).

BRILL (A.). Vorlesungen über algebraische Kurven und algebraische Funktionen. Braunschweig, Vieweg, 1925. 10+340 pp.

CASTELNUOVO (G.). Lezioni di geometria analitica. 5a edizione. Roma, Società Editrice Dante Alighieri, 1922. 8+605 pp.

CIANI (E.). Lezioni di geometria proiettiva ed analitica. 3a edizione, riveduta. Bologna, Zanichelli, 1922. 8+624 pp.

COLSON (A.). See HANOTAUX (G.).

COURANT (R.). See HURWITZ (A.).

CUNNINGHAM (A. J. C.). Binomial factorizations, giving extensive congruence-tables and factorization-tables. Volume 3. London, Francis Hodgson, 1925. 69+203 pp.

CUNNINGHAM (H. E.). Textbook of logic. New York, Macmillan, 1924. 10+428 pp.

ENCYKLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN. Band II 3, Heft 7: L. Zoretti und A. Rosenthal, Die Punktmengen; P. Montel und A. Rosenthal, Integration und Differentiation; M. Fréchet und A. Rosenthal, Funktionenfolgen. Leipzig, Teubner, 1924.

FABRY (C.). See HANOTAUX (G.).

FRÉCHET (M.). See ENCYKLOPÄDIE.

FUHLBERG-HORST (J.). See BLOCH (W.).

HANOTAUX (G.). Histoire de la nation française. Tome 14: Histoire des sciences en France. Volume 1: Introduction générale, par Emile Picard; Mathématiques, mécanique, astronomie, physique et chimie par Henri Andoyer, Pierre Humbert, Charles Fabry, Albert Colson. Paris, Société de l'Histoire Nationale, 1924.

HUMBERT (P.). See HANOTAUX (G.).

HURWITZ (A.) und COURANT (R.). Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. 2te vollständig umgearbeitete und vermehrte Auflage. (Die Grundlehren der mathematischen Wissenschaften, Band 3.) Berlin, Springer, 1925.

KAMKE (E.). Das Lebesguesche Integral. Eine Einführung in die neuere Theorie der reellen Funktionen. Leipzig, Teubner, 1925. 2+153 pp.

LORIA (G). Curve sghembe speciali algebriche e transcendente. Volume I: Curve algebriche. Bologna, Zanichelli, 1925. 12+374 pp.

MESHENBERG (M.P.). Algebraic geometry. A first course, including an introduction to the conic sections. London, Sidgwick and Jackson, 1924. 11+127 pp.

MONTEL (P.). See ENCYKLOPÄDIE.

PICARD (E.). See HANOTAUX (G.).

ROSENTHAL (A.). See ENCYKLOPÄDIE.

RÜCKLE (G.). Praxis des Zahlenrechnens. Charlottenburg, R. Otto Mittelbach, 1925.

RUSSELL (B.). See WHITEHEAD (A.N.).

SCHOUTEN (J. A.). Ueber die Entwicklung der Begriffe des Raumes und der Zeit und ihre Beziehungen zum Relativitätsprinzip. Leipzig, Teubner, 1924. 41 pp.

SCOTT (A. C.). An introductory account of certain modern ideas and methods in plane analytical geometry. 2d edition with notes and corrections. New York, Stechert, 1924. 20+288 pp.

STEVIN (S.). La "Thiende" de Simon Stevin. Facsimile de l'édition originale plantinienne de 1585. Avec une introduction par H. Bosmans, S. J. Anvers, Société des Bibli philes Anversois, 1924. 42+37 pp.

SULLIVAN (J. W. N.). The history of mathematics in Europe from the fall of Greek science to the rise of the conception of mathematical rigour. London and New York, Oxford University Press, 1925. 109 pp. $1.00

TZITZÉICA (G.). Géométrie différentielle projective des réseaux. (Académie Roumaine, Etudes et Recherches, 1.) Paris, Gauthier-Villars, 1924. 3+292 pp.

VALIRON (G.). Fonctions entières et fonctions méromorphes. (Mémorial des Sciences Mathématiques, No. 2.) Paris, Gauthiers-Villars, 1925. 59 pp.

VIVANTI (G.). Elementi del calcolo delle variazioni. Messina, Casa Editrice Giuseppe Principato, 1923. 290 pp.

DE VRIES (H.). Leerboek der differentiaalrekening en van de theorie der differentiaalvergelijkingen. Druk 2. Deel 1. Groningen, Noordhoff, 1924. 765 pp.

WHITEHEAD (A. N.) and RUSSELL (B.). Principia mathematica. 2d edition. Volume 1. Cambridge, University Press, 1925. 46+674 pp.

WHITEHEAD (A. N.). An inquiry concerning the principles of natural knowledge. 2d edition. Cambridge University Press, 1925. 14+207 pp.

WIELEITNER (H.). Die Geburt der modernen Mathematik. Band 2. Karlsruhe, G. Braun, 1925.

ZORETTI (L.). See ENCYKLOPÄDIE.

PART II. APPLIED MATHEMATICS

ACKERMANN (R.). See PEEK (F. W.).

ADAMS (O. S.). Elliptic functions applied to conformal world maps. (U. S. Coast and Geodetic Survey, Special Publication No. 112.) Washington, Government Printing Office, 1925. 4+118 pp.

AIR MINISTRY. Hand book of strength calculations. London, H. M. Stationery Office, 1925.

APRAIZ (F.). L'éther existe et les phénomènes électromagnétiques sont purement mécaniques. Paris, Gauthier-Villars, 1924. 8+40 pp.

BENOIST (L.). See BETHMANN (H.).

BERGET (A.). See CHAPPUIS (J.).

BETHMANN (H.). Les appareils de levage, leur calcul et leur construction. Traduit de la 7e édition allemande par L. Benoist. Paris, Gauthier-Villars, 1925. 784 pp.

BIRTWISTLE (G.). The principles of thermodynamics. Cambridge, University Press, 1925. 10+164 pp.

BOND (W. N.). An introduction to fluid motion. New York, Longmans, 1925. 96 pp. $1.65

BOREL (E.). Traité du calcul des probabilités et de ses applications. Tome 1, fascicule 1: Calcul des probabilités. Tome 2, fascicule 3: Mécanique statistique classique. Paris, Gauthier-Villars, 1925. 10+159+148 pp.

BUCHERER (A. H.). Die Planetenbewegung auf Grund der Quantentheorie und eine Kritik der Einsteinschen Gravitationsgleichungen. 2te Auflage. Bonn, Röhrscheid, 1924. 4+42 pp.

CAILLER (C.). Introduction à la mécanique rationnelle. Ouvrage publié par H. Fehr et R. Wavre. Paris, Gauthiers-Villars, 1924. 12+628 pp.

CHAPPUIS (J.) et BERGET (A.). Leçons de physique générale. Tome 4: Ondes électriques, radioactivité, électro-optique. Paris, Gauthier-Villars, 1925. 4+214 pp.

CHWOLSON (O. D.). Die Physik und ihre Bedeutung für die Menschheit. Aus dem Russischen übersetzt von G. Kluge. Braunschweig, Vieweg, 1924. 8+277 pp.

CZAPSKI (S.) und EPPENSTEIN (O.). Grundzüge der Theorie der optischen Instrumente nach Abbe. 3te Auflage. Leipzig, Barth, 1924. 20+747 pp.

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EWING (J. A.). Thermodynamique. Traduction de M. R. Duchêne. Paris, Gauthier-Villars, 1924. 488 pp.

FEHR (H.). See CAILLER (C.).

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GARDNER (R.). The strength of rotating discs. London, Association of Engineering and Shipbuilding Draughtsmen, 1925.

HOLZINGER (K. J.). Statistical tables for students in education and psychology. Chicago, University of Chicago Press, 1925. 5+74 pp.

JAEGER (F. M.). Le principle de symétrie et ses applications. Paris, Gauthier-Villars, 1925.

DE JANS (C.). Sur la stabilité du mouvement d'une particule massique dans le champ de Schwarzschild. Paris, Gauthier-Villars, 1924. 122 pp.

JONES (R.). See PLANCK (M.).

JORDAN (W.). Handbuch der Vermessungskunde. Fortgesetzt von C. Reinhertz. 7te, erweiterte Auflage. Band 3. Stuttgart, J. B. Metzlersche Verlagsbuchhandlung, 1923. 11+836+79 pp.

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KLUGE (G.). See CHWOLSON (O. D.).

MANES (A.). Versicherungs-Lexikon. Berlin, Mittler, 1924.

PEARL (R.). Studies in human biology. Baltimore, Williams and Wilkins, 1924. 653 pp.

PEEK (F. W.). Phénomènes diélectriques dans la technique des hautes tensions. Traduit sur la seconde édition américaine par R. Ackermann. Paris, Delagrave, 1924. 16+316 pp.

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REINHERTZ (C.). See JORDAN (W.).

TUTTON (A. E. H.). The natural history of crystals. London, Kegan Paul, Trench, Trubner, 1924. 12+287 pp.

UNDERHILL (C. R.). Magnets. New York, McGraw-Hill, 1924. 13 +468 pp.

DE LA VALLÉE POUSSIN (C.). Leçons de mécanique analytique. Tome 1: Vecteurs, cinématique, dynamique du point, statistique. Paris, Gauthier-Villars, 1924. 282 pp.

VERSLUYS (J.). Perspektief. Druk 4. Deel 3. Groningen, Noordhoff, 1924. 150 pp.

VOLLU (L. N.). Recherches sur la physique terrestre; théorie du mouvement des avions. Rio de Janeiro, Besnard, 1924.

WAVRE (R.). See CAILLER (C.).

DE VRIES (H.) en WIJDENES (P.). Leerboek der beschrijvende meetkunde. Deel 2. Groningen, Noordhoff, 1925.

WEINMANN (R.). Anti-Einstein. Leipzig, Hillmann, 1923. 20 pp.

WEISS (E. H.). Traité élémentaire de mécanique. 2 volumes. Paris, Garnier Frères, 1924. 412 pp.

WIJDENES (P.). See DE VRIES (H.).

WILLIAMS (D. H.). See PLANCK (M.).

WOODS (A.). Jule and the study of energy. (Classics of Scientific Method.) London, Bell, 1925. 8+88 pp. + 8 plates.